Constancy of the dimension for $\mathrm{RCD}(K, N)$ spaces via regularity of Lagrangian flows.[1]

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[1] Based on joint works with E. Brué

Plan

Motivations

Theorem (Gromov '81)

The class of n-dimensional Riemannian manifolds with $\mathrm{Ric} \geq K$ and diameter $\leq D$ is precompact w.r.t. the GH metric.

Definition (Ricci limit space)

We say that a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is a Ricci limit space if it is a mGH limit of a sequence of smooth Riemannian manifolds $(M_n, \mathsf{d}_{g_n}, \mathrm{Vol}_n)$ with dimension uniformly bounded from above and $\mathrm{Ric}_n \geq K > -\infty$.

Deep theory initiated by Cheeger-Colding in the nineties.

Can we give a notion of space with Ricci curvature bounded from below without having a smooth structure?

Analogy with the theory of Alexandrov spaces (sectional curvature bounds).

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Background

Definition (Logarithmic entropy)

Given a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, we define the *logarithmic entropy* $\mathrm{Ent}_{\mathfrak{m}} : \mathscr{P}_2(X) \to [-\infty, +\infty]$ by

$$\mathrm{Ent}_{\mathfrak{m}}(\nu) := \begin{cases} \int_X \rho \ln \rho \mathrm{d}\mathfrak{m} & \text{if } \nu = \rho\mathfrak{m} \text{ and } \int_X \rho [\ln \rho]_+ \mathrm{d}\mathfrak{m} < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem (Cordero Erausquin-McCann-Schmuckenschläger '01, Sturm-Von Renesse '05)

On any smooth Riemannian manifold $(M, \mathrm{d}_g, \mathrm{Vol}_g)$ the following are equivalent:

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(ii) $\mathrm{Ric} \geq K$.

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The curvature-dimension condition

Definition (Sturm '06, Lott-Villani '06)

We say that a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{CD}(K, \infty)$ if $\mathrm{Ent}_{\mathfrak{m}}$ is weakly K-convex on $(\mathscr{P}_2(X), W_2)$.

Finite dimensional counterpart: for $N \geq 1$ the Rényi entropy $\mathcal{E}_{N,\mathfrak{m}} : \mathscr{P}_2(X) \to \mathbb{R}$ is defined by

$$\mathcal{E}_{N,\mathfrak{m}}(\mu) = -\int_X \rho^{1-\frac{1}{N}} \mathrm{d}\mathfrak{m}, \quad \text{if } \mu = \rho\mathfrak{m} + \mu^{\perp}.$$

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More generally one can introduce the class of $\mathrm{CD}(K, N)$ spaces, asking for weighted convexity properties of Rényi's entropies on $(\mathscr{P}_2(X), W_2)$, [Sturm '06].

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How to single out Riemannian-like spaces?

On any m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we can introduce the *Cheeger energy* $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathrm{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X |\nabla f_n|^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|_*$ such that $\mathrm{Ch}(f) = \int_X |\nabla f|_*^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathrm{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathrm{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition (Gigli '12)

A metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

On any infinitesimally Hilbertian m.m.s. the heat flow and the Laplacian are linear.

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- equivalent Eulerian approach to the $\mathrm{RCD}(K, N)$ condition based on Bochner's inequality ([Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15]);
- stability and compatibility with the smooth case: weighted Riemannian manifolds and Ricci limits are $\mathrm{RCD}(K, N)$.

Existence and regularity of tangent spaces

Question

How regular is an $\mathrm{RCD}(K,N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ and $x \in X$ we let $Tan_x(X,\mathsf{d},\mathfrak{m})$ be the set of all pmGH limits

$$(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y) = \lim_{i\to\infty}(X, r_i^{-1}\mathsf{d}, \mathfrak{m}^x_{r_i}, x),$$

where $r_i \downarrow 0$ and $\mathfrak{m}^x_{r_i} = c^x_{r_i}\mathfrak{m}$ for some normalizing constant $c^x_{r_i} > 0$.

Theorem (Gigli-Mondino-Rajala '12)

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. Then for $\mathfrak{m}$-a.e. $x \in X$ there exists $1 \le k_x \le N$ such that

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Rectifiability

Definition (k-regular set)

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \le k \le N$ let

$$\mathcal{R}_k := \left\{ x \in X : Tan_x(X, \mathrm{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathrm{d}_{eucl}, \mathscr{L}^k, 0^k \right) \right\} \right\}.$$

Theorem (Mondino-Naber '14)

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k \right) = 0.$$

Furthermore, for any $1 \le k \le N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable.

Theorem (Kell-Mondino '16, Gigli-Pasqualetto '16, De Philippis-Marchese-Rindler '16)

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Constancy of the dimension: Ricci limit spaces

Theorem (Colding-Naber '12)

Let (M^n, g) be a smooth Riemannian manifold with $\mathrm{Ric} \geq K$. Then there exist $C(n, K, \delta) > 0$ and $0 < \alpha(n) < 1$ such that, for any minimizing geodesic $\gamma : [0, 1] \to M$, for any $s, t \in [\delta, 1 - \delta]$ and for any $r \leq 1$, it holds

$$\mathrm{d}_{GH}\left(B_r(\gamma(s)), B_r(\gamma(t))\right) \leq \frac{C}{\delta} |t - s|^{\alpha} r.$$

Theorem (Colding-Naber '12)

Let $(X, \mathrm{d}, \mathfrak{m})$ be a mGH-limit of Riemannian manifolds with $\mathrm{Ric} \geq K$ and dimension less than N. Then there exists $1 \leq n \leq N$ such that

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Theorem (Brué-S. '18)

Let $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. for some* $K \in \mathbb{R}$ *and* $1 \le N < +\infty$*. Then there exists* $1 \le n \le N$ *such that*

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Differences and similarities with [Colding-Naber '12]:

- no smooth approximating sequence;
- we avoid the use of the second order differentiation formula (developed in the RCD framework in [Gigli-Tamanini '17]) and we obtain quantitative estimates that are new also in the smooth setting;
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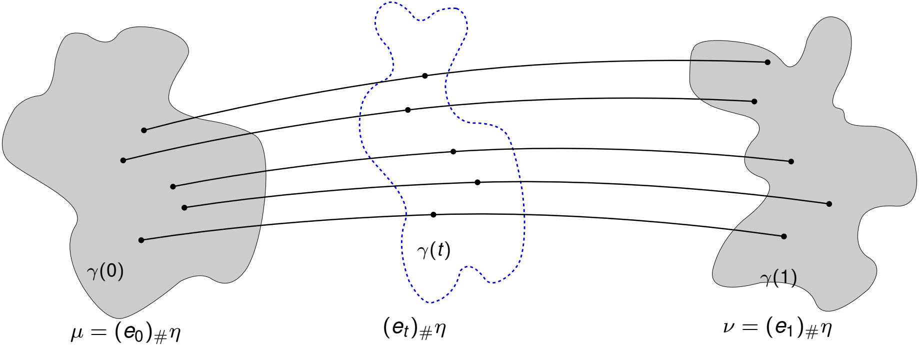
i) Find a “rich enough” class of “regular enough” vector fields and an associated notion of flow;

ii) investigate the regularity of these flows;

iii) prove rigidity statements for maps with the above studied regularity;

iv) combine the ingredients to prove constancy of the dimension.



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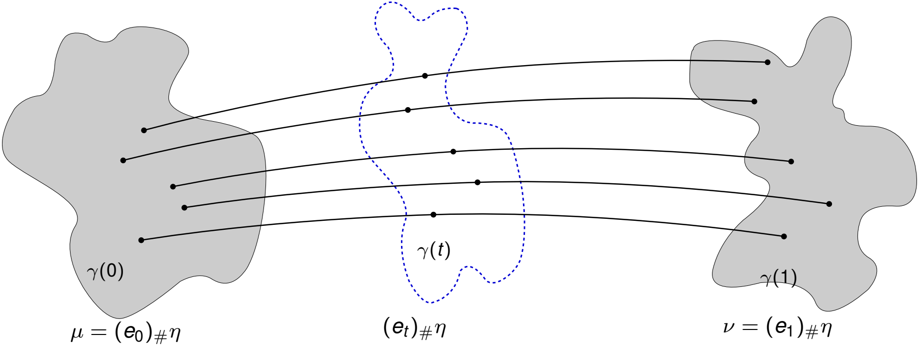

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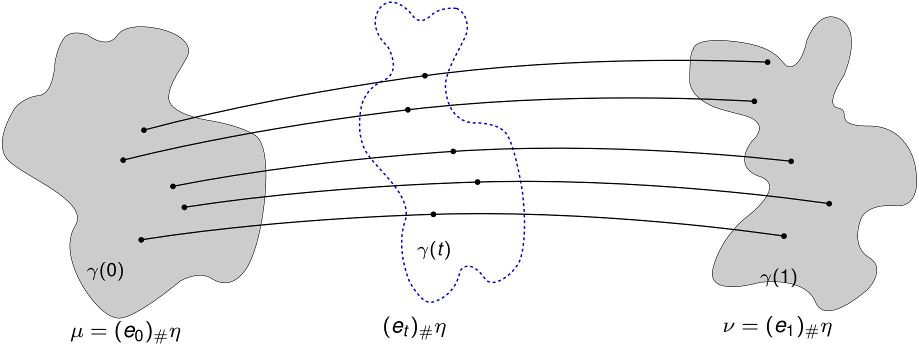

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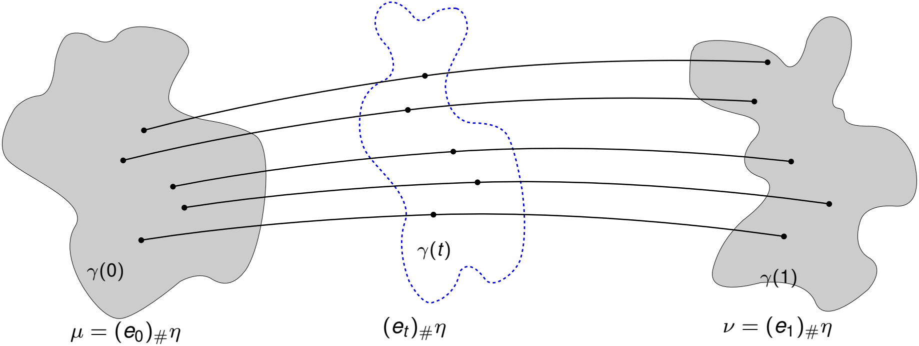

Vector fields on non smooth spaces

On any m.m.s. we consider vector fields as derivations ([Weaver '00]).

Definition (Derivation)

We say that a linear functional $b : \mathrm{Lip}_b(X, \mathsf{d}) \to L^0(X, \mathfrak{m})$ is a *derivation* in $L^2(X, \mathfrak{m})$ if

$$b(fg) = b(f)g + f\,b(g) \quad \forall f, g \in \mathrm{Lip}_b(X, \mathsf{d})$$

and there exists $g \in L^2(X, \mathfrak{m})$ such that

$$|b(f)| \le g|\nabla f|_* \ \mathfrak{m}\text{-a.e. in } X, \text{ for all } f \in \mathrm{Lip}_b(X, \mathsf{d}).$$

We denote by $|b|$ the smallest function in the $\mathfrak{m}$-a.e. sense with this property.

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$$\int_X \boldsymbol{b}(f) \mathrm{d}\mathfrak{m} = -\int_X gf \mathrm{d}\mathfrak{m} \quad \forall f \in \mathrm{Lip}_b(X, \mathrm{d}).$$

Vector fields on non smooth spaces

On any m.m.s. we consider vector fields as derivations ([Weaver '00]).

Definition (Derivation)

We say that a linear functional $\boldsymbol{b} : \mathrm{Lip}_b(X, \mathrm{d}) \to L^0(X, \mathfrak{m})$ is a *derivation* in $L^2(X, \mathfrak{m})$ if

$$\boldsymbol{b}(fg) = \boldsymbol{b}(f)g + f\boldsymbol{b}(g) \quad \forall f, g \in \mathrm{Lip}_b(X, \mathrm{d})$$

and there exists $g \in L^2(X, \mathfrak{m})$ such that

$$|\boldsymbol{b}(f)| \le g|\nabla f|_* \ \mathfrak{m}\text{-a.e. in } X, \text{ for all } f \in \mathrm{Lip}_b(X, \mathrm{d}).$$

We denote by $|\boldsymbol{b}|$ the smallest function in the $\mathfrak{m}$-a.e. sense with this property.

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Sobolev vector fields

Definition

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ m.m.s. and $\boldsymbol{b}$ be a derivation in L^2 with $div\,\boldsymbol{b}$ in L^2. We say that $\nabla_{sym}\boldsymbol{b} \in L^2(X, \mathfrak{m})$ if there exists $c \geq 0$ such that

$$\left| -\frac{1}{2} \int_X \{ \boldsymbol{b}(f)\Delta g + \boldsymbol{b}(g)\Delta f - (div\,\boldsymbol{b})\nabla f \cdot \nabla g)\} \,\mathrm{d}\mathfrak{m} \right| \leq c \| |\nabla f|_* \|_4 \| |\nabla g|_* \|_4,$$

for any $f, g \in \mathrm{Lip}_b(X, \mathrm{d}) \cap D(\Delta)$ such that $|\nabla f|_*, |\nabla g|_* \in L^4(X, \mathfrak{m})$. We let $\|\nabla_{sym}\boldsymbol{b}\|_2$ be the minimal c with such property.

Question

Why do we care about flows of Sobolev vector fields?

- In the Euclidean setting motivations come from the applications to physical problems;
- In the setting of synthetic lower Ricci bounds this is the least regularity we can hope for to have a rich theory.

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Regular Lagrangian flows: the Euclidean case

Definition (Ambrosio '04)

Given a family of vector fields $\boldsymbol{b}_t : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ we say that $\boldsymbol{X}_t : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a *Regular Lagrangian flow* if

a) for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ the curve $t \mapsto \boldsymbol{X}_t(x)$ is an absolutely continuous integral curve of $\boldsymbol{b}_t$;

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Theorem (Di Perna-Lions '89, Ambrosio '04)

Let b_t be a bounded vector field such that

$$b_t \in L^1((0, T); W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}^d)), \quad \operatorname{div} b_t \in L^1((0, T); L^\infty(\mathbb{R}^d)).$$

Then the RLF associated to b_t exists and it is unique: if X and Y are RLF's of b_t, then $X(\cdot, x) = Y(\cdot, x)$ for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$.

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Regularity: the Euclidean case

Remark

Flows of Lipschitz vector fields are Lipschitz.

Theorem (Crippa-De Lellis '08)

Let $\boldsymbol{b}_t \in L^1((0,T); W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ for some $p > 1$ be a bounded vector field. Let $\boldsymbol{X}_t$ be a RLF associated to $\boldsymbol{b}_t$ with compressibility constant $L \geq 0$. Then $\boldsymbol{X}_t$ is Lusin-Lipschitz regular, uniformly w.r.t. time.

Key tools for the proof:

- quantitative Lusin-Lipschitz approximation of scalar valued Sobolev functions (extends to metric setting);
- possibility of dealing with vector fields componentwise (no analogue in the curved/metric setting).

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Regular Lagrangian flows on $\mathrm{RCD}(K, \infty)$ spaces

Definition

Given a family of derivations $\boldsymbol{b}_t$ on $[0, T]$ we say that $\boldsymbol{X} : [0, T] \times X \to X$ is a Regular Lagrangian flow if

a) $\boldsymbol{X}(\cdot, x) \in \mathrm{AC}([0, T]; X)$ for $\mathfrak{m}$-a.e. $x \in X$;

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Regularity in the non smooth setting 1

We say that a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *n-Ahlfors* regular for some $n > 0$ if there exist $C > 1$ and $R > 0$ such that

$$\frac{1}{C} r^n \leq \mathfrak{m}(B(x, r)) \leq C r^n \quad \text{for any } x \in X \text{ and } 0 < r < R.$$

Theorem (Bruè-S. '18)

Let $(X, \mathrm{d}, \mathfrak{m})$ be a compact, n-Ahlfors regular $\mathrm{RCD}(K, N)$ m.m.s.. Assume that

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The result applies to Alexandrov spaces and non collapsed $\mathrm{RCD}(K, N)$ spaces (as defined in [Gigli-De Philippis '17]).

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$$\frac{1}{C} r^n \leq \mathfrak{m}(B(x, r)) \leq C r^n \quad \text{for any } x \in X \text{ and } 0 < r < R.$$

Theorem (Bruè-S. '18)

Let $(X, \mathrm{d}, \mathfrak{m})$ be a compact, n-Ahlfors regular $\mathrm{RCD}(K, N)$ m.m.s.. Assume that

$$\boldsymbol{b} \in L^\infty(X, \mathfrak{m}), \quad \|\nabla_{sym} \boldsymbol{b}_t\|_2 \in L^2(0, T), \quad |div \boldsymbol{b}_t| \in L^1((0, T); L^\infty(X, \mathfrak{m})).$$

Then the RLF $\boldsymbol{X}_t$ of $\boldsymbol{b}_t$ is Lusin Lipschitz regular, uniformly w.r.t. time.

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The result applies to Alexandrov spaces and non collapsed $\mathrm{RCD}(K, N)$ spaces (as defined in [Gigli-De Philippis '17]).

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Remark

A key tool for the proof of the regularity result in the Ahlfors regular case is the Green function of the Laplacian (G such that $\Delta G = -\delta$).

Definition

An $\mathrm{RCD}(0, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is said to be *non parabolic* if there exists a positive Green function of the Laplacian.
If there exists $x_0 \in X$ such that

$$\int_1^{+\infty} \frac{r}{\mathfrak{m}(B(x_0, r))} \mathrm{d}r < +\infty,$$

then the space is non parabolic.

Theorem

Let $(X, \mathsf{d}, \mathfrak{m})$ be a non parabolic $\mathrm{RCD}(0, N)$ m.m.s.. Then (X, d_G) is a quasi metric space, where $\mathsf{d}_G := 1/G$. If an additional volume growth condition is satisfied, $(X, \mathsf{d}_G, \mathfrak{m})$ is doubling.

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Let $(X, \mathrm{d}, \mathfrak{m})$ be as above and let $\boldsymbol{b}_t$ be a compactly supported bounded vector field with bounded divergence such that $\|\nabla_{sym} \boldsymbol{b}_t\|_2 \in L^1(0, T)$. Let $\boldsymbol{X}_t$ be the RLF of $\boldsymbol{b}_t$. Then $\boldsymbol{X}_t$ is Lusin Lipschitz regular w.r.t. d_G, uniformly w.r.t. time.

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How to extend the results to arbitrary $\mathrm{RCD}(K, N)$ spaces?

- Change $(X, \mathrm{d}, \mathfrak{m})$ into $(X, \mathrm{d}, \mathfrak{m}) \times (\mathbb{R}^k, \mathrm{d}_{eucl}, \mathscr{L}^k)$ to gain the volume growth conditions;
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From regularity to constant dimension 1

Formal statement 1: transitivity

If there exist $k \neq n$ such that $\mathfrak{m}(\mathcal{R}_k), \mathfrak{m}(\mathcal{R}_n) > 0$, then we can find probability measures $\mu_1 \ll \mathfrak{m} \llcorner \mathcal{R}_k$ and $\mu_2 \ll \mathfrak{m} \llcorner \mathcal{R}_n$ and a "regular" vector field $\boldsymbol{b}$ with RLF $\boldsymbol{X}$ such that $(\boldsymbol{X}_t)_\sharp \mu_1 = \mu_2$ for some $t > 0$

Key tools for the proof:

- Lewy-Stampacchia inequality in the non smooth setting ([Gigli-Mosconi '15]);
- the inclusion $D(\Delta) \subset W^{2,2}(X, \mathsf{d}, \mathfrak{m})$ holds on any $\mathrm{RCD}(K, \infty)$ m.m.s. ([Gigli '14]).

Bilipschitz maps preserve the Hausdorff dimension.
Is the regularity of RLFs compatible with existence of $\mathrm{RCD}(K, N)$ spaces with non constant dimension?

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Remark

The Green function combines in intrinsic way distance and measure.

Formal statement 2: asymptotic behaviour

Near to k-regular points the Green function behaves like the Green function on $\mathbb{R}^k$.

Formal statement 2 and an argument a la Sard, together with the known results about structure theory yield:

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A d_G-Lusin regular map with bounded compressibility cannot map a piece of dimension n into a piece of dimension $k < n$.

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Consequences

Theorem (Bruè-S. '18)

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Given the constancy of dimension for $\mathrm{RCD}(K, N)$ spaces the following sounds natural:

Definition

Given an $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we let $\dim X$ be the unique natural $1 \leq n \leq N$ such that $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$.

Theorem (Kitabeppu '18)

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then, for any* $k > \dim X$*, it holds* $\mathcal{R}_k = \emptyset$*.*

Open problems

Conjecture

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ it holds that

$$\dim X = \dim_H X.$$

Furthermore, for any natural $n \neq \dim X$ it holds $\mathcal{R}_n = \emptyset$.

The conjecture above is open also for Ricci limit spaces.

Question

Can we prove a statement analogous to the Hölder continuity of tangent cones by Colding-Naber in the setting of $\mathrm{RCD}(K,N)$ spaces, possibly using the regularity theory of Lagrangian flows?

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