Overview of RCD spaces II

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Outline

Splitting theorem

Theorem ([Gigli '13])

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(0, N)$ space. Suppose it contains a line. Then it splits as $\mathbb{R} \times Y$ in the sense of metric measure spaces, where $(Y, \mathrm{d}_Y, \mathrm{m}_Y)$ is an $\mathrm{RCD}(0, N-1)$ space.

- [Cheeger-Gromoll '72] for smooth Riemannian manifolds;
- [Cheeger-Colding '96] for Ricci limit spaces;
- [Gigli '13] for $\mathrm{RCD}(0, N)$ spaces.

Remark

With the current RCD technology a proof very much in the spirit of [Cheeger-Gromoll '72] can be given.

Volume cone implies metric cone

Given a metric space (Y, d) with $\mathrm{diam}(Y) \leq \pi$ the cone distance on $C(Y) := (0, +\infty) \times Y$ is defined by

$$\mathrm{d}^2_{C(Y)}((r_1, y_1), (r_2, y_2)) := r_1^2 + r_2^2 - 2r_1 r_2 \cos(\mathrm{d}(y_1, y_2)) .$$

The N-metric measure cone over $(Y, \mathrm{d}, \mathrm{m})$ is endowed with the measure $r^{N-1}\mathrm{d}r \otimes \mathrm{m}$. See [Ketterer '15] for important results on cones in RCD setting.

Theorem ([De Philippis-Gigli '16] after [Cheeger-Colding '96])

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(0, N)$ space and let $p \in X$ and $R > 0$ be such that

$$\frac{\mathrm{m}(B_R(p))}{R^N} = \frac{\mathrm{m}(B_{2R}(p))}{(2R)^N} .$$

Then there exists an $\mathrm{RCD}(N-2, N-1)$ space $(Y, \mathrm{d}_Y, \mathrm{m}_Y)$ such that the ball $B_R(p)$ is isomorphic to the ball centred at the vertex of the cone $C(Y)$.

Bochner inequality with Hessian term

In the most recent developments of the RCD theory a fundamental role is played by the Bochner inequality with Hessian term:

$$\Delta\frac{1}{2}|\nabla u|^2 \geq ||\mathrm{Hess}\, u||^2_{\mathrm{HS}} + \nabla u \cdot \nabla \Delta u + K|\nabla u|^2 \,,$$

for sufficiently regular functions $u : X \to \mathbb{R}$ on an $\mathrm{RCD}(K,\infty)$ space $(X, \mathrm{d}, \mathrm{m})$.

The inequality is proved in [Gigli '14] building on top of [Savaré '13], [Ambrosio-Gigli-Savaré '11, '12]. See also [Sturm '14] and [Bakry '85].

Almost rigidity theorems

Almost rigidity theorems can be proved by combining compactness, stability of the RCD condition and the rigidity theorems.

Theorem (Almost splitting, rough statement)

An $\mathrm{RCD}(-\varepsilon, N)$ space which contains a very long segment is close to a product in the pointed measured Gromov-Hausdorff sense.

Theorem (Almost v.c. implies almost m.c., rough statement)

If on an $\mathrm{RCD}(-\varepsilon, N)$ space the Bishop-Gromov volume ratio is almost constant between $R = 1$ and $R = 2$ at some point p, then $B_1(p)$ is close to the ball in a cone in the mGH-sense.

Tangent cones and lower Ricci bounds

Remark

Tangent cones are unique and metric cones on Alexandrov spaces.

Highly false in general under lower Ricci curvature bounds:

- first examples of non uniqueness of tangent cones at infinity for $\mathrm{Ric} \geq 0$ and Euclidean volume growth in [Perelman '97];
- several examples of degenerate behaviours in [Cheeger-Colding '97], in particular tangent cones that are not metric cones;
- three dimensional manifolds with Euclidean volume growth $\mathrm{Ric} \geq 0$ and non unique blow-downs in [Colding-Naber '13].

Existence of Euclidean tangents

Remark

At an interior point of a geodesic any tangent cone splits a line, by the splitting theorem.

The iterative application of the splitting theorem yields:

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(K, N)$ space. Then for m-a.e. $x \in X$ there exists $1 \leq k \leq N$ such that some tangent space at x is k-dimensional Euclidean.

Rectifiability

Theorem ([Burago-Gromov-Perelman '92])

On an Alexandrov space one can find a biLipschitz chart in a neighbourhood of a point with Euclidean tangent cone.

Theorem ([Mondino-Naber '14])

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(K, N)$ space. Then:

- *the tangent cone is unique for m-a.e. point $x \in X$;*
- *the set $\mathcal{R}_k$ of points with tangent cone k-dimensional Euclidean is (m, k)-rectifiable.*

The analogous statement for Ricci limit spaces was proved in [Cheeger-Colding '97], with different proof.

Behaviour of the reference measure

Question

What about the behaviour of the reference measure m?

Building on top of [De Philippis-Rindler '16], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16] proved:

Theorem

Let $(X, \mathrm{d}, \mathrm{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then*

$$\mathrm{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k \llcorner \mathcal{R}_k \,,$$

for every $1 \leq k \leq N$*, for some locally integrable non negative density* θ*.*

The analogous statement was obtained for Ricci limit spaces in [Cheeger-Colding '00].

Constancy of dimension

In [Cheeger-Colding '00] uniqueness of the rectifiable dimension for Ricci limit spaces was conjectured.

Theorem

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(K, N)$ space. There exists $1 \le n \le N$, called essential dimension, such that tangent cones are n-dimensional Euclidean at m-a.e. point in X.

- [Colding-Naber '12] for Ricci limit spaces;
- [Bruè-S. '18] for RCD spaces with different proof;
- proof in the spirit of [Colding-Naber '12] in [Deng '20].

Continuity of tangent cones

On Alexandrov spaces tangent cones are constant in the interior of minimizing geodesics [Petrunin '98].

Theorem ([Deng '20] after [Colding-Naber '12])

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $\gamma : [0, 1] \to X$ be a geodesic. Then balls $B_r(\gamma(t))$ along the interior of γ change in scale invariantly Hölder continuous way in GH-sense.

Corollary

Tangent cones along the same sequence of scalings are Hölder continuous in GH-sense in the interior of geodesics.

Corollary ([Deng '20])

$\mathrm{RCD}(K, N)$ *spaces are non branching.*

Remarks and questions

Remark

The essential dimension of an $\mathrm{RCD}(K, N)$ space $(X, \mathrm{d}, \mathrm{m})$ might be strictly less than the Hausdorff dimension of (X, d), [Pan-Wei, '21].

Remark

Geodesics on $\mathrm{RCD}(K, N)$ spaces might branch instantaneously, i.e. they are not determined by their velocities, [Colding-Naber, '13].

Open question

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(K, N)$ space. Do there exist points $x \in X$ with a neighbourhood homeomorphic to a topological manifold?

Motivations

Ricci limit spaces are much better understood under the non collapsing assumption:

$$(M_i^N, d_i, p_i) \to (X, \mathrm{d}, p)\,,$$

$$(M_i^N, d_i, p_i)\,, \quad \mathrm{Ric}_i \geq K\,, \quad \mathrm{vol}(B_1(p_i)) > v > 0\,.$$

Theorem ([Colding '97], [Cheeger-Colding '97])

Under the assumptions above $\dim_{\mathcal{H}}(X, \mathrm{d}) = N$ *and*

$$(M_i, d_i, \mathrm{vol}_i, p_i) \to (X, \mathrm{d}, \mathcal{H}^N, p)$$

in the pointed mGH-sense.

Terminology

$\mathrm{RCD}(K, N)$ spaces $(X, \mathrm{d}, \mathrm{m})$ for which $\mathrm{m} = \mathcal{H}^N$ are called *non collapsed* or *N-dimensional*.

Theorem ([Brena-Gigli-Honda-Zhu '21] after [Honda '18])

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(K, N)$ space. Then the essential dimension n equals N if and only if $\mathrm{m} = \mathcal{H}^N$ up to multiplicative constants.

In this class

$$\Delta u = \mathrm{tr}\,(\mathrm{Hess}\, u)$$

for sufficiently regular functions $u : X \to \mathbb{R}$. The identity fails on (non trivially) weighted Riemannian manifolds $(M, g, e^{-V}\mathrm{vol}_g)$.

First results

For $\mathrm{RCD}(K, N)$ spaces $(X, \mathrm{d}, \mathcal{H}^N)$ [De Philippis-Gigli '18] proved:

- a volume convergence theorem holds;
- all iterated tangent cones are metric cones;
- if one tangent cone is $\mathbb{R}^N$ at some point, then it is unique;
- setting $\mathcal{S}^k \subset X$ the set of those points where no tangent cone splits $\mathbb{R}^{k+1}$, it holds

$$\dim_{\mathcal{H}}(\mathcal{S}^k) \leq k\,.$$

Interior topological regularity

Theorem ([Kapovitch-Mondino '19] after [Cheeger-Colding '97])

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. Then an open neighbourhood of the regular set is biHölder homeomorphic to a smooth Riemannian manifold.

Remark

The proof builds on a variant of Reifenberg's theorem valid for metric spaces and proved in [Cheeger-Colding '97].

Earlier insights on topological regularity under Ricci curvature bounds come from [Anderson '90] and [Perelman '94].

Harmonic *coordinates*

Theorem ([Bruè-Naber-S. '20] after [Cheeger-Jiang-Naber '18])

For any $0 < \alpha < 1$ there exists $\varepsilon(\alpha, N) > 0$ such that if $(X, \mathrm{d}, \mathcal{H}^N)$ is an $\mathrm{RCD}(-\varepsilon, N)$ space and

$$\mathrm{d}_{GH}(B_2(p), B_2(0^N)) < \varepsilon \,,$$

then there exists a map $u : B_{3/2}(p) \to \mathbb{R}^N$ such that

- *u has harmonic (hence Lipschitz) components;*
- *u is a α-biHölder homeomorphism with its image;*
- *$B_1(0^N) \subset u(B_1(p)) \subset \mathbb{R}^N$.*

Remark

The construction of harmonic almost splitting maps under lower Ricci bounds in smooth setting goes back to [Cheeger-Colding '96].

Interior topological singularities

Example

The Eguchi-Hanson metric is a Ricci flat metric on the cotangent bundle of $\mathbb{S}^2$ with Euclidean volume growth. Its blow-down is $\mathbb{R}^4/\mathbb{Z}_2$, which is not a topological manifold near to the origin.

Example

The cone over $\mathbb{RP}^2$ with standard metric is an $\mathrm{RCD}(0,3)$ space with a topological singularity at the vertex.

Boundary regularity and stability

Definition

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. We define the boundary ∂X of X as the topological closure of the top dimensional singular stratum:

$$\partial X := \overline{\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}}\,.$$

Theorem ([Bruè-Naber-S. '20])

The boundary of an $\mathrm{RCD}(K, N)$ space $(X, \mathrm{d}, \mathcal{H}^N)$ is $(N-1)$-rectifiable with locally finite $\mathcal{H}^{N-1}$-measure.
Tangent cones are unique and isometric to Euclidean half-spaces at any point in $\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}$.

Theorem ([Bruè-Naber-S. '20] after [Cheeger-Colding '97])

Non collapsed limits of $\mathrm{RCD}(K, N)$ spaces $(X_i, \mathrm{d}_i, \mathcal{H}^N)$ with empty boundary have empty boundary.

Topological boundary regularity

Theorem ([Bruè-Naber-S. '20])

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. Then there exists an open set $O \subset X$ with

$$\dim_{\mathcal{H}}(X \setminus O) \leq N - 2$$

such that O is biHölder homeomorphic to a smooth manifold, possibly with boundary.
Moreover, $\partial X \cap O$ is homeomorphic to an $(N-1)$-dimensional manifold without boundary.

Remark

The key step for establishing boundary regularity is proving that closeness to a half-space in GH-sense implies existence of *many* boundary points.

Open questions

Conjecture [Anderson-Cheeger-Colding-Tian]

Non collapsed Ricci limit spaces are topological manifolds away from sets of codimension four.

Theorem ([Simon-Topping '17])

Any three dimensional non collapsed Ricci limit space is a topological manifold.

Conjecture [Kapovitch-Mondino '19]

Any $\mathrm{RCD}(K,N)$ space $(X,\mathrm{d},\mathcal{H}^N)$ with empty boundary is a topological manifold away from a set of codimension three.

Alexandrov vs RCD

Remark

In the smooth case lower sectional curvature bounds clearly imply lower Ricci curvature bounds.
In dimension two Ricci curvature and sectional curvature coincide.

Theorem ([Petrunin '09] see also [Zhang-Zhu '09])

Let (X, d) be an Alexandrov space of dimension n with curvature bounded from below by k. Then $(X, \mathrm{d}, \mathcal{H}^n)$ is an $\mathrm{RCD}(k(n-1), n)$ metric measure space.

Theorem ([Lytchak-Stadler '18])

Let $(X, \mathrm{d}, \mathcal{H}^2)$ be an $\mathrm{RCD}(K, 2)$ space. Then (X, d) is an Alexandrov space with sectional curvature bounded from below by K.

Open questions: scalar curvature

Theorem ([Petrunin '08])

There exists a constant $C(n) > 0$ such that for any smooth Riemannian manifold (M^n, g) with sectional curvature bounded below by -1 it holds

$$\frac{1}{\mathrm{vol}(B_1(p))} \int_{B_1(p)} |\mathrm{Scal}| \,\mathrm{dvol} \le C(n)\,.$$

Conjecture [Yau '90]

An analogous integral bound holds if the lower sectional curvature bound is replaced by a lower Ricci curvature bound.

Open question: scalar curvature

Open question [Petrunin-Lebedeva '22]

Find a (well behaved) notion of scalar curvature measure on Alexandrov spaces. More in general of measure-valued curvature.

Open question

Find a notion of scalar curvature measure for $\mathrm{RCD}(K, N)$ spaces $(X, \mathrm{d}, \mathcal{H}^N)$ consistent with the smooth case and with the Gauss-Bonnet theorem when $N = 2$.

See [Kapovitch-Lytchak-Petrunin '17] and [Gigli '19] for related results/proposals.

Thank you for your attention!