Geometric Measure Theory on non smooth spaces with lower Ricci bounds

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Introduction

Question

Do area minimizing hypersurfaces in non smooth spaces with lower curvature bounds have vanishing mean curvature?
If yes, in which sense?

Does the lower Ricci curvature bound affect the second variation of the area in non smooth ambient spaces? If yes, in which sense?

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Outline

Ricci bounds and minimal hypersurfaces, 1

Consider a smooth Riemannian manifold (M^n, g) and a smooth compact codimension one hypersurface $\Sigma^{n-1} \subset M$.
Consider a compactly supported smooth vector field X and denote its flow by $\Phi_t : M \times (-\varepsilon, \varepsilon) \to M$. Then we have the first variation formula

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_\Sigma \mathrm{div}_\Sigma X \,\mathrm{d}\,\mathcal{H}^{n-1} = -\int_\Sigma H \cdot X \,\mathrm{d}\,\mathcal{H}^{n-1} .$$

Corollary

If Σ minimizes the area among compactly supported perturbations then it is minimal, i.e. $H = 0$.

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Ricci bounds and minimaly hypersurfaces, 2

If Σ is minimal and two-sided with unit normal ν, then we can compute the second variation of the area for vector fields X such that $X = f\nu$ along Σ:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int \left[|\nabla_\Sigma f|^2 - \left(|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu)f^2\right)\right] \mathrm{d}\,\mathcal{H}^{n-1}$$

Theorem (Simons, Ann. of Math. '68)

There are no closed two sided stable minimal hypersurfaces in a closed manifold with positive Ricci curvature.

Neglecting the regularity issues:

Corollary

There is no two sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.

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Why GMT with non smooth lower Ricci bounds

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and Euclidean volume growth, i.e.

$$\lim_{r \to \infty} \frac{\mathrm{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1] .$$

Then we can consider:

- pointed limits at infinity, i.e. limits in the pGH topology of sequences $(M, \mathrm{d}_g, \mathcal{H}^n, p_i)$, where $\mathrm{d}_g(p_i, p) \to \infty$;
- blow-downs, i.e. limits in the pGH topology of sequences $(M, \mathrm{d}_g/r_i, \mathcal{H}^n/r_i^n, p)$, where $r_i \to \infty$.

In both cases we get non collapsed Ricci limit spaces $(X, \mathrm{d}, \mathcal{H}^n)$, that are not smooth in general.

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RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
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Why the synthetic perspective?

The quest for a synthetic notion of lower Ricci curvature bounds was evident since the early developments of the theory [Gromov '91], [Cheeger-Colding '97].

Remark

There are statements which were open for Ricci limit spaces that have been proved with the synthetic approach.

Theorem (Deng '20)

Any RCD(K, N) metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is metrically non branching. All the minimizing geodesics of a Ricci limit space are limit geodesics.

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Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

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- Motivations
- **Non smooth lower Ricci bounds**
- Main results
- Extensions

Let us consider an $\mathrm{RCD}(K, N)$ space $(X, \mathrm{d}, \mathcal{H}^N)$. We assume that it has no boundary for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

- (X, d) is N-rectifiable [Mondino-Naber, *JEMS* '14];
- (X, d) is bi-Hölder homeomorphic to a smooth manifold away from a set of codimension two [Kapovitch-Mondino, *Geom. Topol.* '21] after [Cheeger-Colding '97];
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- (X, d) is bi-Hölder homeomorphic to a smooth manifold away from a set of codimension two [Kapovitch-Mondino, *Geom. Topol* '21] after [Cheeger-Colding '97];
- all tangent cones are metric cones and there is a stratification of the singular set [De Philippis-Gigli, *J. Éc. polytech. Math.* '18] after [Cheeger-Colding '97].

Regularity of non collapsed RCD spaces

Let us consider an $\mathrm{RCD}(K, N)$ space $(X, \mathrm{d}, \mathcal{H}^N)$. We assume that it has no boundary for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

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Bad examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathsf{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) \geq v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathsf{d}_Y, y)$ where

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Mean curvature: examples

Example

Let D be a flat two dimensional disk with boundary C. Let $\tilde{D}$ be the metric space obtained by doubling D along the boundary.

Given the current knowledge of RCD spaces:

- a classical regularity theory for area minimizing hypersurfaces does not make sense;
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Area minimizing hypersurfaces

We deal with sets of finite perimeter.

- Their Euclidean theory was pioneered by Caccioppoli and De Giorgi in the Fifties;
- the theory on metric measure spaces was initiated in [Ambrosio, Adv. Math. '02];
- the theory on RCD(K, N) spaces is well understood (perimeter equals codimension one measure, rectifiability with well defined dimension, Gauss-Green formula) after [Ambrosio-Bruè-S., GAFA '19], [Bruè-Pasqualetto-S. '19, '21]

Definition

Let $(X, \mathsf{d}, \mathfrak{m})$ be an RCD(K, N) space. We say that $E \subset \Omega \subset X$ is a local perimeter minimizer if for any $x \in E$ there is a neighbourhood $U_x \ni x$ such that

$$\mathrm{Per}(E, U_x) \le \mathrm{Per}(F, U_x), \quad \text{if } F\Delta E \Subset U_x.$$

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

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Remarks

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The regularity theory for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Local perimeter minimizers have open representatives whose topological boundary coincides with the measure theoretic boundary, [Kinnunen et al., *J. Geom. Anal.* '13].

Local perimeter minimality is a very general condition. All smooth minimal hypersurfaces are locally perimeter minimizing boundaries, locally.

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Main result

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)}\tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
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Theorem (Mondino–S. '21)

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a local perimeter minimizer. Let $\mathsf{d}_{\overline{E}} : X \setminus \overline{E} \to [0, \infty)$ be the distance function from $\overline{E}$. Then

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Sharpness and remarks

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $\mathrm{RCD}(K, N)$ spaces, they can be understood distributionally;
- the bounds are sharp and attained on the model spaces;
- on $\mathbb{R}^n$ the bounds imply that ∂E is a viscosity solution of the minimal surfaces equation [Savin, Comm. PDEs '07];
- the bounds carry the sharp information about the first and second variation of the area of equidistant hypersurfaces. When everything is smooth, they give vanishing mean curvature;
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- the bounds make perfectly sense on $\mathrm{RCD}(K, N)$ spaces, they can be understood distributionally;
- the bounds are sharp and attained on the model spaces;
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GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Corollaries

The strict super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the Gauss-Green theorem imply:

Theorem (Generalized Simons' theorem)

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(N-1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

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Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(N-1, N)$ space. Let $\Sigma_1, \Sigma_2 \subset X$ be closed sets that are locally boundaries of locally perimeter minimizing sets. Then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$.

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The classical case

If $\partial E = \Sigma \subset X$ is smooth and minimal inside a smooth Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the viscosity sense.

- Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) mean curvature;
- the information is propagated along minimizing geodesics up to the cut-locus via a Jacobi fields computation;
- the contribution coming from singularities of the distance function has the right sign.

Together with [Calabi, *Duke Math. J.* '57] and [Cheeger-Gromoll, *JDG* '71] this was a motivation for a theory of super-harmonic functions in weak sense.

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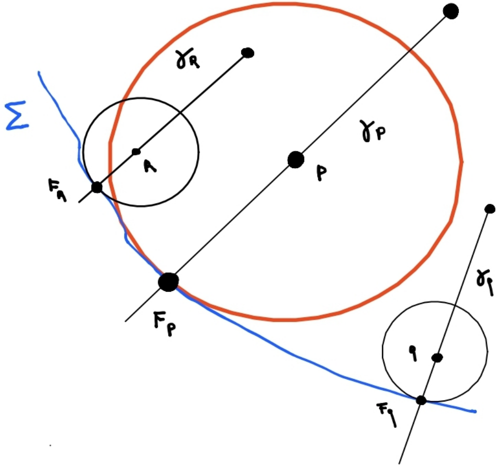
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Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on $\mathbb{R}^n$ and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

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It exploits the duality between viscous, distributional and variational interpretation of Laplacian bounds.

- in [Caffarelli-Cordoba '93] the bound on the Laplacian is proved via the viscosity theory, using comparison with quadratic polynomials and the affine structure.
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- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a lower supporting function φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - \mathsf{d}(y, z)\} .$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \le \mathsf{d}_E$.
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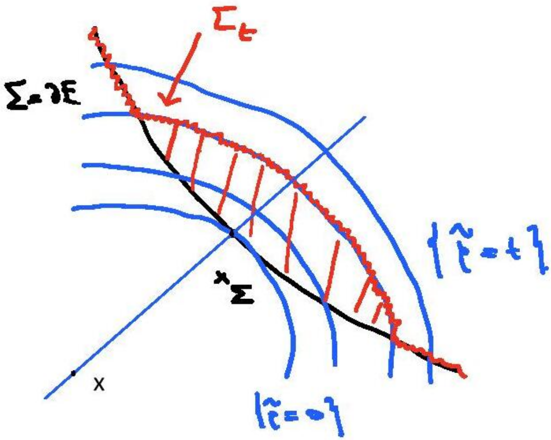

The PDE principle

Theorem (Mondino-S. '21)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(0, N)$ space and let $\varphi : X \to \mathbb{R}$ be super-harmonic. Then

$$Q_t\varphi(x) := \inf_{y\in X}\{\varphi(y) + \frac{\mathrm{d}^p(x,y)}{pt^{p-1}}\}$$

is super-harmonic for any $1 \le p \le \infty$ and for any $t > 0$.

- The proof completely avoids parallel transport and Jacobi fields computations.
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The isoperimetric problem

Remark

In the classical setting, area minimizing hypersurfaces have *vanishing mean curvature* and isoperimetric sets have *constant mean curvature*.

Is there any extension of the theory for isoperimetric sets?

In [Antonelli-Pasqualetto-Pozzetta-S. '21] we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\mathrm{RCD}(K, N)$ space $(X, \mathrm{d}, \mathcal{H}^N)$.

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Consequences

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Thank you for your attention!