

Geometric Measure Theory on non smooth spaces with lower Ricci bounds

Daniele Semola

Mathematical Institute, University of Oxford

Daniele.Semola@maths.ox.ac.uk

Geometric Analysis Seminar - UC San Diego

Introduction

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Do area minimizing hypersurfaces in non smooth spaces with lower curvature bounds have vanishing mean curvature?
If yes, in which sense?

Does the lower Ricci curvature bound affect the volume of the area of the area in non smooth ambient spaces? If yes, in which sense?

Introduction

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Question

Do **area minimizing** hypersurfaces in non smooth spaces with lower curvature bounds have **vanishing mean curvature**?
If yes, in which sense?

Does the lower Ricci curvature bound affect the second variation of the area in non smooth ambient spaces? If yes, in which sense?

Introduction

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Question

Do **area minimizing** hypersurfaces in non smooth spaces with lower curvature bounds have **vanishing mean curvature**?
If yes, in which sense?

Does the lower Ricci curvature bound affect the second variation of the area in non smooth ambient spaces? If yes, in which sense?

Introduction

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Question

Do **area minimizing** hypersurfaces in non smooth spaces with lower curvature bounds have **vanishing mean curvature**?
If yes, in which sense?

Question

Does the **lower Ricci** curvature bound affect the **second variation** of the area in non smooth ambient spaces? If yes, in which sense?

Outline

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

1 Motivations

2 Non smooth lower Ricci bounds

3 Main results

4 Extensions

Ricci bounds and minimal hypersurfaces, 1

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Consider a smooth Riemannian manifold (M^n, g) and a smooth compact codimension one hypersurface $\Sigma^{n-1} \subset M$.

Consider a compactly supported smooth vector field X and denote its flow by $\Phi_t : M \times (-\varepsilon, \varepsilon) \rightarrow M$. Then we have the first variation formula

$$\frac{d}{dt} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} \operatorname{div}_{\Sigma} X \, d\mathcal{H}^{n-1} = - \int_{\Sigma} H \cdot X \, d\mathcal{H}^{n-1}.$$

Ricci bounds and minimal hypersurfaces, 1

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Consider a **smooth** Riemannian manifold (M^n, g) and a smooth compact **codimension one** hypersurface $\Sigma^{n-1} \subset M$.

Consider a compactly supported smooth vector field X and denote its **flow** by $\Phi_t : M \times (-\varepsilon, \varepsilon) \rightarrow M$. Then we have the **first variation formula**

$$\frac{d}{dt} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} \operatorname{div}_{\Sigma} X \, d\mathcal{H}^{n-1} = - \int_{\Sigma} H \cdot X \, d\mathcal{H}^{n-1}.$$

If Σ minimizes the area among compactly supported perturbations then it is minimal, i.e. $H \equiv 0$.

Ricci bounds and minimal hypersurfaces, 1

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Consider a **smooth** Riemannian manifold (M^n, g) and a smooth compact **codimension one** hypersurface $\Sigma^{n-1} \subset M$.

Consider a compactly supported smooth vector field X and denote its **flow** by $\Phi_t : M \times (-\varepsilon, \varepsilon) \rightarrow M$. Then we have the **first variation formula**

$$\frac{d}{dt} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} \operatorname{div}_{\Sigma} X \, d\mathcal{H}^{n-1} = - \int_{\Sigma} H \cdot X \, d\mathcal{H}^{n-1}.$$

If Σ minimizes the area among compactly supported perturbations then it is minimal, i.e. $H \equiv 0$.

Ricci bounds and minimal hypersurfaces, 1

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Consider a **smooth** Riemannian manifold (M^n, g) and a smooth compact **codimension one** hypersurface $\Sigma^{n-1} \subset M$.

Consider a compactly supported smooth vector field X and denote its **flow** by $\Phi_t : M \times (-\varepsilon, \varepsilon) \rightarrow M$. Then we have the **first variation formula**

$$\frac{d}{dt} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} \operatorname{div}_{\Sigma} X \, d\mathcal{H}^{n-1} = - \int_{\Sigma} H \cdot X \, d\mathcal{H}^{n-1}.$$

Corollary

If Σ **minimizes the area** among compactly supported perturbations then it is **minimal**, i.e. $H \equiv 0$.

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is minimal and two-sided with unit normal ν , then we can compute the second variation of the area for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\mathbb{H}|^2 + \text{Ric}(\nu, \nu)f^2)] d\mathcal{H}^{n-1}$$

When applied to closed two-sided surfaces, it provides a lower bound on the second variation with positive Ricci curvature.

Neglecting the regularity issues:

There is no lower bound on the second variation of the area for a closed surface with positive Ricci curvature.

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is **minimal** and **two-sided** with unit normal ν , then we can compute the **second variation of the area** for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\text{II}|^2 + \text{Ric}(\nu, \nu)f^2)] d\mathcal{H}^{n-1}$$

There are no closed two sided stable minimal hypersurfaces in a closed manifold with positive Ricci curvature.

Neglecting the regularity issues:

The second variation of the area is non-negative for all vector fields $X = f\nu$ along Σ if and only if $\text{Ric}(\nu, \nu) \geq |\text{II}|^2$.

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is **minimal** and **two-sided** with unit normal ν , then we can compute the **second variation of the area** for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\mathbf{II}|^2 + \text{Ric}(\nu, \nu)f^2)] \, d\mathcal{H}^{n-1}$$

There are no closed two sided stable minimal hypersurfaces in a closed manifold with positive Ricci curvature.

Neglecting the regularity issues:

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is **minimal** and **two-sided** with unit normal ν , then we can compute the **second variation of the area** for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\mathbb{II}|^2 + \text{Ric}(\nu, \nu)f^2)] d\mathcal{H}^{n-1}$$

Theorem (Simons *Ann. of Math.* '68)

*There are no closed two sided **stable** minimal hypersurfaces in a closed manifold with positive Ricci curvature.*

Neglecting the regularity issues:

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is **minimal** and **two-sided** with unit normal ν , then we can compute the **second variation of the area** for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\mathbb{II}|^2 + \text{Ric}(\nu, \nu)f^2)] d\mathcal{H}^{n-1}$$

Theorem (Simons *Ann. of Math.* '68)

*There are no closed two sided **stable** minimal hypersurfaces in a closed manifold with positive Ricci curvature.*

Neglecting the regularity issues:

There is no two sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is **minimal** and **two-sided** with unit normal ν , then we can compute the **second variation of the area** for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\mathbb{II}|^2 + \text{Ric}(\nu, \nu)f^2)] d\mathcal{H}^{n-1}$$

Theorem (Simons *Ann. of Math.* '68)

*There are no closed two sided **stable** minimal hypersurfaces in a closed manifold with positive Ricci curvature.*

Neglecting the regularity issues:

There is no two sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.

Ricci bounds and minimally hypersurfaces, 2

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If Σ is **minimal** and **two-sided** with unit normal ν , then we can compute the **second variation of the area** for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int [|\nabla_{\Sigma} f|^2 - (|\mathbb{I}|^2 + \text{Ric}(\nu, \nu)f^2)] d\mathcal{H}^{n-1}$$

Theorem (Simons *Ann. of Math.* '68)

*There are no closed two sided **stable** minimal hypersurfaces in a closed manifold with positive Ricci curvature.*

Neglecting the regularity issues:

Corollary

There is no two sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.

Why GMT with non smooth lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and **Euclidean volume growth**, i.e.

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1].$$

Then we can consider:

- **pointed limits at infinity**, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g(p_i, p) \rightarrow \infty$;
- **pointed limits at infinity**, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g \rightarrow \infty$.

In both cases we get **non collapsed Ricci limit spaces** (X, d, \mathcal{H}^n) , that are not smooth in general.

Why GMT with non smooth lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and **Euclidean volume growth**, i.e.

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1].$$

Then we can consider:

- pointed limits at infinity, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g(p_i, p) \rightarrow \infty$,
- blow-downs, i.e. limits in the pGH topology of sequences $(M, d_g/r_i, \mathcal{H}^n/r_i^n, p)$, where $r_i \rightarrow \infty$.

In both cases we get **non collapsed Ricci limit spaces** (X, d, \mathcal{H}^n) , that are not smooth in general.

Why GMT with non smooth lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and **Euclidean volume growth**, i.e.

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1].$$

Then we can consider:

- **pointed limits** at infinity, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g(p_i, p) \rightarrow \infty$;
- **blow-downs**, i.e. limits in the pGH topology of sequences $(M, d_g/r_i, \mathcal{H}^n/r_i^n, p)$, where $r_i \rightarrow \infty$.

In both cases we get **non collapsed Ricci limit spaces** (X, d, \mathcal{H}^n) , that are not smooth in general.

Why GMT with non smooth lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and **Euclidean volume growth**, i.e.

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1].$$

Then we can consider:

- **pointed limits** at infinity, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g(p_i, p) \rightarrow \infty$;
- **blow-downs**, i.e. limits in the pGH topology of sequences $(M, d_g/r_i, \mathcal{H}^n/r_i^n, p)$, where $r_i \rightarrow \infty$.

In both cases we get **non collapsed Ricci limit spaces** (X, d, \mathcal{H}^n) , that are not smooth in general.

Why GMT with non smooth lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and **Euclidean volume growth**, i.e.

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1].$$

Then we can consider:

- **pointed limits** at infinity, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g(p_i, p) \rightarrow \infty$;
- **blow-downs**, i.e. limits in the pGH topology of sequences $(M, d_g/r_i, \mathcal{H}^n/r_i^n, p)$, where $r_i \rightarrow \infty$.

In both cases we get **non collapsed Ricci limit spaces** (X, d, \mathcal{H}^n) , that are not smooth in general.

Why GMT with non smooth lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider a smooth open manifold (M^n, g) with non negative Ricci curvature and **Euclidean volume growth**, i.e.

$$\lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(p))}{\omega_n r^n} = \theta \in (0, 1].$$

Then we can consider:

- **pointed limits** at infinity, i.e. limits in the pGH topology of sequences $(M, d_g, \mathcal{H}^n, p_i)$, where $d_g(p_i, p) \rightarrow \infty$;
- **blow-downs**, i.e. limits in the pGH topology of sequences $(M, d_g/r_i, \mathcal{H}^n/r_i^n, p)$, where $r_i \rightarrow \infty$.

In both cases we get **non collapsed Ricci limit** spaces (X, d, \mathcal{H}^n) , that are not smooth in general.

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

$RCD(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $RCD(K, N)$;

- $RCD(K, N)$ spaces are closed under taking limits;

- $RCD(K, N)$ spaces are closed under taking products;

- the space $RCD(K, N)$ is closed under taking quotients;

- $RCD(K, N)$ spaces are closed under taking covers;

- $RCD(K, N)$ spaces are closed under taking blow-ups;

- $RCD(K, N)$ spaces are closed under taking blow-downs;

- $RCD(K, N)$ spaces are closed under taking rescalings;

- $RCD(K, N)$ spaces are closed under taking dilations;

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

$\text{RCD}(K, N)$ metric measure spaces (X, d, \mathfrak{m}) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

• Ricci limits are $\text{RCD}(K, N)$;

• Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is closed under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15];
- when $m = \mathcal{H}^N$ they are much more regular [Cheeger-Colding '97], [De Philippis-Gigli '18], [Kapovitch-Mondino-21].

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is closed under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15];
- when $m = \mathcal{H}^N$ they are much more regular [Cheeger-Colding '97], [De Philippis-Gigli '18], [Kapovitch-Mondino-21].

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is closed under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15];
- when $m = \mathcal{H}^N$ they are much more regular [Cheeger-Colding '97], [De Philippis-Gigli '18], [Kapovitch-Mondino-21].

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15];
- when $m = \mathcal{H}^N$ they are much more regular [Cheeger-Colding '97], [De Philippis-Gigli '18], [Kapovitch-Mondino-21].

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15];
- when $m = \mathcal{H}^N$ they are much more regular [Cheeger-Colding '97], [De Philippis-Gigli '18], [Kapovitch-Mondino-21].

RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15];
- when $m = \mathcal{H}^N$ they are much more regular [Cheeger-Colding '97], [De Philippis-Gigli '18], [Kapovitch-Mondino-21].

Why the synthetic perspective?

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The quest for a **synthetic** notion of lower Ricci curvature bounds was evident since the early developments of the theory [Gromov '91], [Cheeger-Colding '97].

There are statements which were open for Ricci limit spaces that have been proved with the synthetic approach.

For a fixed $K, \nu \in \mathbb{R}$, a metric measure space (X, d, μ) is said to have (K, ν) -RCD if the minimizing geodesics of a Ricci limit space are

Why the synthetic perspective?

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The quest for a **synthetic** notion of lower Ricci curvature bounds was evident since the early developments of the theory [**Gromov '91**], [**Cheeger-Colding '97**].

There are statements which were open for Ricci limit spaces that have been proved with the synthetic approach.

Why the synthetic perspective?

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The quest for a **synthetic** notion of lower Ricci curvature bounds was evident since the early developments of the theory [Gromov '91], [Cheeger-Colding '97].

Remark

There are statements which were open for Ricci limit spaces that have been proved with the synthetic approach.

Any $\text{RCD}(K, N)$ metric measure space (X, d, m) is metrically non-branching. All the minimizing geodesics of a Ricci limit space are limit geodesics.

Why the synthetic perspective?

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The quest for a **synthetic** notion of lower Ricci curvature bounds was evident since the early developments of the theory [Gromov '91], [Cheeger-Colding '97].

Remark

There are statements which were open for Ricci limit spaces that have been proved with the synthetic approach.

Any $\text{RCD}(K, N)$ metric measure space (X, d, m) is metrically non-branching. All the minimizing geodesics of a Ricci limit space are limit geodesics.

Why the synthetic perspective?

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The quest for a **synthetic** notion of lower Ricci curvature bounds was evident since the early developments of the theory [Gromov '91], [Cheeger-Colding '97].

Remark

There are statements which were open for Ricci limit spaces that have been proved with the synthetic approach.

Theorem (Deng '20)

*Any $\text{RCD}(K, N)$ metric measure space (X, d, m) is metrically **non branching**. All the minimizing geodesics of a Ricci limit space are **limit geodesics**.*

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Brue-Naber-S. '20], [Cheeger-Colding, JDG '97].

Then:

- (X, d) is a complete metric space with bounded diameter
- (X, d) is homeomorphic to a smooth manifold
- (X, d) is a Riemannian manifold with bounded sectional curvature
- (X, d) is a Riemannian manifold with bounded Ricci curvature
- (X, d) is a Riemannian manifold with bounded scalar curvature

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Brue-Naber-S. '20], [Cheeger-Colding, JDG '97].

Then:

Regularity of non collapsed RCD spaces

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

• (X, d) is N -rectifiable [Mondino-Naber, *JEMS* '14];

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

- (X, d) is N -rectifiable [Mondino-Naber, *JEMS* '14];
- (X, d) is α -Holder homeomorphic to a smooth manifold away from a set of codimension two [Kapovitch-Mondino, *Geom. Topol* '21] after [Cheeger-Colding '97];

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

- (X, d) is **N -rectifiable** [Mondino-Naber, *JEMS* '14];
- (X, d) is **bi-Hölder** homeomorphic to a smooth manifold away from a set of **codimension two** [Kapovitch-Mondino, *Geom. Topol* '21] after [Cheeger-Colding '97];
- all tangent cones are **metric cones** and there is a **stratification** of the singular set [De Philippis-Gigli, *J. Éc. polytech. Math.* '18] after [Cheeger-Colding '97].

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

- (X, d) is **N -rectifiable** [Mondino-Naber, *JEMS* '14];
- (X, d) is **bi-Hölder** homeomorphic to a smooth manifold away from a set of **codimension two** [Kapovitch-Mondino, *Geom. Topol* '21] after [Cheeger-Colding '97];
- all tangent cones are **metric cones** and there is a **stratification** of the singular set [De Philippis-Gigli, *J. Éc. polytech. Math.* '18] after [Cheeger-Colding '97].

Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let us consider an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) . We assume that it has no **boundary** for simplicity.

Remark

The condition $\partial X = \emptyset$ is stable under non collapsed GH convergence [Bruè-Naber-S. '20], [Cheeger-Colding, *JDG* '97].

Then:

- (X, d) is **N -rectifiable** [Mondino-Naber, *JEMS* '14];
- (X, d) is **bi-Hölder** homeomorphic to a smooth manifold away from a set of **codimension two** [Kapovitch-Mondino, *Geom. Topol* '21] after [Cheeger-Colding '97];
- all tangent cones are **metric cones** and there is a **stratification** of the singular set [De Philippis-Gigli, *J. Éc. polytech. Math.* '18] after [Cheeger-Colding '97].

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

There exist (M_n^g, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, \rho_n) \rightarrow (\mathbb{R}^2, d_{\text{Eucl}}, 0)$$

but

$$\text{inrad}(\rho_n) \rightarrow 0.$$

Let $\rho_n \in \mathcal{P}(\mathbb{R}^2)$ be a probability measure on \mathbb{R}^2 with $\text{supp}(\rho_n) = \{0, 1, \dots, n\}$ and $\int_{\mathbb{R}^2} |x|^2 d\rho_n \geq 0$ and $(M_n, g_n, \rho_n) \rightarrow (\mathbb{R}^2, d_{\text{Eucl}}, 0)$.

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{eucl}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > \nu > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{eucl}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > \nu > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{\text{eucl}}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have **Euclidean tangent cone**;
- d_Y is **not** induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are **not well defined** at y .

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{\text{eucl}}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have **Euclidean tangent cone**;
- d_Y is **not** induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are **not well defined** at y .

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{\text{eucl}}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have **Euclidean tangent cone**;
- d_Y is **not** induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y .

Bad examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{\text{eucl}}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have **Euclidean tangent cone**;
- d_Y is **not** induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are **not well defined** at y .

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions



Given the current knowledge of RCD spaces:

- the classical rigidity theory for non-minimising hypersurfaces does not make sense
- whether the first variation formula for the area can be extended to non-minimising hypersurfaces remains an open question
- the rigidity theory for hypersurfaces

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$ (there is singular distributional Gaussian curvature along the copy of C ;

Given the current knowledge of RCD spaces:

is $(\tilde{D}, d_{\tilde{D}})$ a Riemannian manifold with bounded curvature?

What is the mean curvature of C ?

What is the mean curvature of the copy of C in \tilde{D} ?

What is the mean curvature of the copy of C in \tilde{D} ?

What is the mean curvature of the copy of C in \tilde{D} ?

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$ (there is singular distributional Gaussian curvature along the copy of C);
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

- the singular part of the curvature is concentrated on the copy of C ;
 - the area functional is not differentiable along the normal variations.
- Can we extend the RCD theory to spaces with singularities?

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$. There is singular distributional Gaussian curvature along the copy of C ;
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$. There is singular distributional Gaussian curvature along the copy of C ;
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$. There is singular distributional Gaussian curvature along the copy of C ;
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

- a classical regularity theory for area minimizing hypersurfaces does not make sense;

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$. There is singular distributional Gaussian curvature along the copy of C ;
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

- a classical regularity theory for area minimizing hypersurfaces does not make sense;
- neither the first variation formula for the area, nor a distributional notion of mean curvature make sense along a prescribed codimension one hypersurface.

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$. There is singular distributional Gaussian curvature along the copy of C ;
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

- a classical regularity theory for area minimizing hypersurfaces does not make sense;
- neither the first variation formula for the area, nor a distributional notion of mean curvature make sense along a prescribed codimension one hypersurface.

Mean curvature: examples

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Example

Let D be a flat two dimensional disk with boundary C . Let \tilde{D} be the metric space obtained by doubling D along the boundary.

- The metric space $(\tilde{D}, d_{\tilde{D}})$ is $\text{RCD}(-1, 2)$. There is singular distributional Gaussian curvature along the copy of C ;
- the area functional is not differentiable along the normal variations.

Given the current knowledge of RCD spaces:

- a classical regularity theory for area minimizing hypersurfaces does not make sense;
- neither the first variation formula for the area, nor a distributional notion of mean curvature make sense along a prescribed codimension one hypersurface.

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with sets of finite perimeter.

- Their Euclidean theory was pioneered by Caccioppoli and De Giorgi in the Fifties;

• the theory on metric measure spaces was initiated in

- the theory of (GDA, μ) spaces is well understood; G is a complete metric space, μ is a Radon measure with well defined dimension, μ is lower Ricci bounded.

Let (X, d, μ) be a metric measure space. We say that (X, d, μ) is a (GDA, μ) space if (X, d) is a complete metric space, μ is a Radon measure with well defined dimension, μ is lower Ricci bounded.

Let F be a half ball $B_{1/2}^+(0) \subset \mathbb{R}^n$.

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with **sets of finite perimeter**.

- Their Euclidean theory was pioneered by Caccioppoli and De Giorgi in the Fifties;
- the theory on metric measure spaces was initiated in [Ambrosio, *Adv. Math.* '02];

The theory of $\text{CDD}(\mu, \theta)$ spaces is well understood in the Euclidean setting, but the theory in the metric measure setting is still under development.

• In the Euclidean setting, the theory of sets of finite perimeter is well understood, but the theory in the metric measure setting is still under development.

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with **sets of finite perimeter**.

- Their Euclidean theory was pioneered by **Caccioppoli** and **De Giorgi** in the Fifties;
- the theory on metric measure spaces was initiated in [**Ambrosio**, *Adv. Math.* '02];
- the theory on $\text{RCD}(K, N)$ spaces is well understood (perimeter equals **codimension one** measure, **rectifiability** with well defined dimension, **Gauss-Green** formula) after [**Ambrosio-Bruè-S.**, *GAFN* '19], [**Bruè-Pasqualetto-S.** '19, '21].

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with **sets of finite perimeter**.

- Their Euclidean theory was pioneered by **Caccioppoli** and **De Giorgi** in the Fifties;
- the theory on metric measure spaces was initiated in **[Ambrosio, *Adv. Math.* '02]**;
- the theory on $RCD(K, N)$ spaces is well understood (perimeter equals **codimension one** measure, **rectifiability** with well defined dimension, **Gauss-Green** formula) after **[Ambrosio-Bruè-S., *GAFN* '19], [Bruè-Pasqualetto-S. '19, '21]**.

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with **sets of finite perimeter**.

- Their Euclidean theory was pioneered by **Caccioppoli** and **De Giorgi** in the Fifties;
- the theory on metric measure spaces was initiated in **[Ambrosio, *Adv. Math.* '02]**;
- the theory on $\text{RCD}(K, N)$ spaces is well understood (perimeter equals **codimension one** measure, **rectifiability** with well defined dimension, **Gauss-Green** formula) after **[Ambrosio-Bruè-S., *GAFSA* '19], [Bruè-Pasqualetto-S. '19, '21]**.

Let (X, d, m) be an $\text{RCD}(K, N)$ space. We say that $E \subset \Omega \subset X$ is a local perimeter minimizer if for any $x \in E$ there is a neighbourhood $U_x \ni x$ such that

$$\text{Per}(E, U_x) \leq \text{Per}(F, U_x) \quad \text{if } F \Delta E \Subset U_x.$$

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with **sets of finite perimeter**.

- Their Euclidean theory was pioneered by **Caccioppoli** and **De Giorgi** in the Fifties;
- the theory on metric measure spaces was initiated in **[Ambrosio, *Adv. Math.* '02]**;
- the theory on $\text{RCD}(K, N)$ spaces is well understood (perimeter equals **codimension one** measure, **rectifiability** with well defined dimension, **Gauss-Green** formula) after **[Ambrosio-Bruè-S., *GAFSA* '19], [Bruè-Pasqualetto-S. '19, '21]**.

Let (X, d, m) be an $\text{RCD}(K, N)$ space. We say that $E \subset \Omega \subset X$ is a local perimeter minimizer if for any $x \in E$ there is a neighbourhood $U_x \ni x$ such that

$$\text{Per}(E, U_x) \leq \text{Per}(F, U_x) \quad \text{if } F \Delta E \Subset U_x.$$

Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

We deal with **sets of finite perimeter**.

- Their Euclidean theory was pioneered by **Caccioppoli** and **De Giorgi** in the Fifties;
- the theory on metric measure spaces was initiated in **[Ambrosio, *Adv. Math.* '02]**;
- the theory on $\text{RCD}(K, N)$ spaces is well understood (perimeter equals **codimension one** measure, **rectifiability** with well defined dimension, **Gauss-Green** formula) after **[Ambrosio-Bruè-S., *GAFN* '19], [Bruè-Pasqualetto-S. '19, '21]**.

Definition

Let (X, d, m) be an $\text{RCD}(K, N)$ space. We say that $E \subset \Omega \subset X$ is a **local perimeter minimizer** if for any $x \in E$ there is a neighbourhood $U_x \ni x$ such that

$$\text{Per}(E, U_x) \leq \text{Per}(F, U_x), \quad \text{if } F \Delta E \Subset U_x.$$

Remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The regularity theory for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Local perimeter minimizers have open neighborhoods whose boundary coincides with the mean curvature flow starting from the initial boundary.

Local perimeter minimizers satisfy a very general condition. All smooth hypersurfaces locally minimize are particular angle.

Remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Remark

The **regularity theory** for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Local perimeter minimizers have open representatives whose topological boundary coincides with the measure theoretic boundary, [Kinnunen et al., *J. Geom. Anal.* '13].

Remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Remark

The **regularity theory** for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Local perimeter minimizers have open representatives whose topological boundary coincides with the measure theoretic boundary, [Kinnunen et al., *J. Geom. Anal.* '13].

Remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Remark

The [regularity theory](#) for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Remark

Local perimeter minimizers have open representatives whose [topological](#) boundary coincides with the [measure theoretic](#) boundary, [[Kinnunen et al., J. Geom. Anal. '13](#)].

Local perimeter minimality is a very general condition. All smooth minimal hypersurfaces are locally perimeter minimizing boundaries, locally.

Remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Remark

The [regularity theory](#) for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Remark

Local perimeter minimizers have open representatives whose [topological](#) boundary coincides with the [measure theoretic](#) boundary, [[Kinnunen et al., J. Geom. Anal. '13](#)].

Local perimeter minimality is a very general condition. All smooth minimal hypersurfaces are locally perimeter minimizing boundaries, locally.

Remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Remark

The [regularity theory](#) for sets of finite perimeter was the starting point for the regularity theory in codimension one in GMT.

Remark

Local perimeter minimizers have open representatives whose [topological](#) boundary coincides with the [measure theoretic](#) boundary, [[Kinnunen et al., J. Geom. Anal. '13](#)].

Remark

Local perimeter minimality is a very general condition. All [smooth minimal](#) hypersurfaces are locally perimeter minimizing boundaries, locally.

Main result

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

$$\mathbb{R}^n_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x) \text{ if } K > 0;$$

$$\mathbb{R}^n_{0,N} := 0;$$

$$\mathbb{R}^n_{K,N} := \sqrt{-K(N-1)} \coth(\sqrt{-K/(N-1)}x) \text{ if } K < 0.$$

Let (M, g) be an (N, K) -RCD space-time and let $\mathbb{R}^n_{K,N}$ be the corresponding Riccati function. Let $\mathcal{R}^n_{K,N}$ be the set of Riccati fields on M and let $\mathcal{R}^n_{K,N}(x)$ be the set of Riccati fields at $x \in M$. Let $\mathcal{R}^n_{K,N}(x) \rightarrow \mathbb{R}^n_{K,N}$ be the natural projection map. Then

$$\mathcal{R}^n_{K,N}(x) \subseteq \mathbb{R}^n_{K,N} \text{ for any } x \in M.$$

Main result

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Main result

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Main result

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a local perimeter minimizer. Let $d_E : X \setminus \bar{E} \rightarrow [0, \infty)$ be the distance function from \bar{E} . Then

$$\Delta d_E \leq \tau_{K,N} \circ d_E \quad \text{on } X \setminus \bar{E}.$$

Main result

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a local perimeter minimizer. Let $d_E : X \setminus \bar{E} \rightarrow [0, \infty)$ be the distance function from \bar{E} . Then

$$\Delta d_E \leq \tau_{K,N} \circ d_E \quad \text{on } X \setminus \bar{E}.$$

Main result

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Theorem (Mondino-S. '21)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a *local perimeter minimizer*. Let $d_{\bar{E}} : X \setminus \bar{E} \rightarrow [0, \infty)$ be the *distance function* from \bar{E} . Then

$$\Delta d_{\bar{E}} \leq \tau_{K,N} \circ d_{\bar{E}} \quad \text{on } X \setminus \bar{E}.$$

Sharpness and remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $\text{RCD}(K, N)$ spaces, they can be understood distributionally;
- the bounds are sharp and attained on the Euclidean space;
- in $\text{RCD}(K, N)$ the bounds imply that N is a singular extension of the Euclidean space;
- the bounds carry the sharpness from the Euclidean space to the more general setting of the area of non-smooth metric measure spaces, when everything is smooth, they give vanishing mean curvature;
- there is no need to use the full machinery of the area of non-smooth metric measure spaces.

Sharpness and remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
 - the bounds make perfectly sense on $RCD(K, N)$ spaces, they can be understood **distributionally**;
 - the bounds are **sharp** and attained on the model spaces;
 - on \mathbb{R}^n the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces equation** [Savin, *Comm. PDEs '07*];
 - the bounds carry the sharp information about the first and second variation of the area of **equidistant hypersurfaces**. When everything is smooth, they give vanishing mean curvature;
 - there is no need to talk about **mean curvature** of the area minimizing boundary;

Sharpness and remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $RCD(K, N)$ spaces, they can be understood **distributionally**;
- the bounds are **sharp** and attained on the model spaces;
- on \mathbb{R}^n the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces equation** [Savin, *Comm. PDEs '07*];
- the bounds carry the sharp information about the first and second variation of the area of **equidistant hypersurfaces**. When everything is smooth, they give vanishing mean curvature;
- there is no need to talk about **mean curvature** of the area minimizing boundary;

Sharpness and remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $RCD(K, N)$ spaces, they can be understood **distributionally**;
- the bounds are **sharp** and attained on the model spaces;
- on \mathbb{R}^n the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces equation** [Savin, *Comm. PDEs '07*];
- the bounds carry the sharp information about the first and second variation of the area of **equidistant hypersurfaces**. When everything is smooth, they give vanishing mean curvature;
- there is no need to talk about **mean curvature** of the area minimizing boundary;

Sharpness and remarks

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $RCD(K, N)$ spaces, they can be understood **distributionally**;
- the bounds are **sharp** and attained on the model spaces;
- on \mathbb{R}^n the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces equation** [Savin, *Comm. PDEs '07*];
- the bounds carry the sharp information about the first and second variation of the area of **equidistant hypersurfaces**. When everything is smooth, they give vanishing mean curvature;
- there is no need to talk about **mean curvature** of the area minimizing boundary;

Sharpness and remarks

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $RCD(K, N)$ spaces, they can be understood **distributionally**;
- the bounds are **sharp** and attained on the model spaces;
- on \mathbb{R}^n the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces equation** [Savin, *Comm. PDEs '07*];
- the bounds carry the sharp information about the first and second variation of the area of **equidistant hypersurfaces**. When everything is smooth, they give vanishing mean curvature;
- there is no need to talk about **mean curvature** of the area minimizing boundary;

Sharpness and remarks

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $\text{RCD}(K, N)$ spaces, they can be understood **distributionally**;
- the bounds are **sharp** and attained on the model spaces;
- on \mathbb{R}^n the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces equation** [Savin, *Comm. PDEs '07*];
- the bounds carry the sharp information about the first and second variation of the area of **equidistant hypersurfaces**. When everything is smooth, they give vanishing mean curvature;
- there is no need to talk about **mean curvature** of the area minimizing boundary;

Corollaries

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **Gauss-Green** theorem imply:

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N-1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **strong maximum principle** imply:

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N-1, N)$ space. Let $T_1, T_2 \subset X$ be closed sets that are locally equivalent to locally convex domains in \mathbb{R}^N . Then $\text{Per}(T_1) \leq \text{Per}(T_2)$ implies $\text{Vol}(T_1) \leq \text{Vol}(T_2)$.

Corollaries

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **Gauss-Green** theorem imply:

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N-1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **strong maximum principle** imply:

Corollaries

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **Gauss-Green** theorem imply:

Theorem (Generalized Simons' theorem)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **strong maximum principle** imply:

Corollaries

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **Gauss-Green** theorem imply:

Theorem (Generalized Simons' theorem)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **strong maximum principle** imply:

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Let $\Sigma_1, \Sigma_2 \subset X$ be closed sets that are locally boundaries of locally perimeter minimizing sets. Then $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Corollaries

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **Gauss-Green** theorem imply:

Theorem (Generalized Simons' theorem)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **strong maximum principle** imply:

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Let $\Sigma_1, \Sigma_2 \subset X$ be closed sets that are locally boundaries of locally perimeter minimizing sets. Then $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Corollaries

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **Gauss-Green** theorem imply:

Theorem (Generalized Simons' theorem)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Then there exists no set $E \subset X$ which minimizes the perimeter among perturbations compactly contained in $B_r(\partial E)$ for some $r > 0$.

The **strict** super-harmonicity of the distance function from a local perimeter minimizer when $K > 0$ and the **strong maximum principle** imply:

Theorem (Generalized Frankel's property)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(N - 1, N)$ space. Let $\Sigma_1, \Sigma_2 \subset X$ be closed sets that are locally boundaries of locally perimeter minimizing sets. Then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$.

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

- Along the minimal boundary the Laplacian of the distance function corresponds to the (vanishing) mean curvature;
- the information is propagated along minimizing geodesics up to the cut-locus via a Hopf-Cole's computation;
- the contribution coming from singularities of the distance function has the right sign.

When the boundary is not smooth, the theory of viscosity solutions is needed.

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) mean curvature;

- the information is propagated along minimizing geodesics up to the cut-locus via a Jacobi fields computation;

Along the boundary the Laplacian of the distance corresponds to the (vanishing) mean curvature;

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

- Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) **mean curvature**;
- the information is **propagated** along minimizing geodesics up to the cut-locus via a **Jacobi fields** computation;
- the contribution coming from singularities of the distance function has the right sign.

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

- Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) **mean curvature**;
- the information is **propagated** along minimizing geodesics up to the cut-locus via a **Jacobi fields** computation;
- the contribution coming from singularities of the distance function has the right sign.

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

- Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) **mean curvature**;
- the information is **propagated** along minimizing geodesics up to the cut-locus via a **Jacobi fields** computation;
- the contribution coming from singularities of the distance function has the right sign.

Together with Delzant, then also of [ST] and [Chever, *Ann Inst. Fourier* '71], this was a motivation for a theory of super-harmonic functions in weak sense.

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

- Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) **mean curvature**;
- the information is **propagated** along minimizing geodesics up to the cut-locus via a **Jacobi fields** computation;
- the contribution coming from singularities of the distance function has the right sign.

Together with Delzant, then also of [ST] and [Chever, *Ann Inst. Fourier* '71], this was a motivation for a theory of super-harmonic functions in weak sense.

The classical case

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

If $\partial E = \Sigma \subset X$ is **smooth** and minimal inside a **smooth** Riemannian manifold then the statement goes back at least to [Wu, *Acta Math.* '79], where the bound was understood in the **viscosity** sense.

- Along the minimal boundary the Laplacian of the distance corresponds to the (vanishing) **mean curvature**;
- the information is **propagated** along minimizing geodesics up to the cut-locus via a **Jacobi fields** computation;
- the contribution coming from singularities of the distance function has the right sign.

Remark

Together with [Calabi, *Duke Math. J.* '57] and [Cheeger-Gromoll, *JDG* '71] this was a motivation for a theory of super-harmonic functions in *weak sense*.

For general area minimizing hypersurfaces (currents, sets of finite perimeter) the classical argument is due to [Gromov '81], where the isoperimetric variant was considered.

- minimality needed only at footpoints F_P of geodesics γ_P on the boundary $\Sigma = \partial E$;
- for a.s. point of the support on the boundary all the blow-ups of the current are contained in a half-space;
- by Allard's regularity theorem the area-minimizing boundary is smooth in a neighborhood of these points;
- then the smooth argument carries over.

For general **area minimizing** hypersurfaces (currents, sets of finite perimeter) the classical argument is due to **[Gromov '81]**, where the **isoperimetric** variant was considered.

- minimality needed only at footpoints F_γ of geodesics γ_γ on the boundary $\Sigma = \partial E$;
- for a.e. point at the footpoint on the boundary all the blow-ups of the current are contained in a half-space;
- the key idea is to use the isoperimetric inequality in the blow-up; this is done by using a regularity estimate of the isoperimetric problem; the isoperimetric problem is solved by a smooth surface; the isoperimetric inequality is used to compare the current with the smooth surface.

For general **area minimizing** hypersurfaces (currents, sets of finite perimeter) the classical argument is due to **[Gromov '81]**, where the **isoperimetric** variant was considered.

- minimality needed only at **footpoints** F_P of geodesics γ_P on the boundary $\Sigma = \partial E$;
- for a.e. point at the **footpoint** on the boundary all the blow-ups of the current are contained in a half-space;
- by **Almgren's regularity theorem** the area minimizing boundary is **smooth** in a neighbourhood of these points;
- then the smooth argument carries over.

For general **area minimizing** hypersurfaces (currents, sets of finite perimeter) the classical argument is due to **[Gromov '81]**, where the **isoperimetric** variant was considered.

- minimality needed only at **footpoints** F_P of geodesics γ_P on the boundary $\Sigma = \partial E$;
- for a.e. point at the **footpoint** on the boundary all the blow-ups of the current are contained in a half-space;
- by **Almgren's regularity theorem** the area minimizing boundary is **smooth** in a neighbourhood of these points;
- then the smooth argument carries over.

For general **area minimizing** hypersurfaces (currents, sets of finite perimeter) the classical argument is due to **[Gromov '81]**, where the **isoperimetric** variant was considered.

- minimality needed only at **footpoints** F_P of geodesics γ_P on the boundary $\Sigma = \partial E$;
- for a.e. point at the **footpoint** on the boundary all the blow-ups of the current are contained in a half-space;
- by **Almgren's regularity theorem** the area minimizing boundary is **smooth** in a neighbourhood of these points;
- then the smooth argument carries over.

For general **area minimizing** hypersurfaces (currents, sets of finite perimeter) the classical argument is due to **[Gromov '81]**, where the **isoperimetric** variant was considered.

- minimality needed only at **footpoints** F_P of geodesics γ_P on the boundary $\Sigma = \partial E$;
- for a.e. point at the **footpoint** on the boundary all the blow-ups of the current are contained in a half-space;
- by **Almgren's regularity theorem** the area minimizing boundary is **smooth** in a neighbourhood of these points;
- then the smooth argument carries over.

GMT and lower Ricci bounds

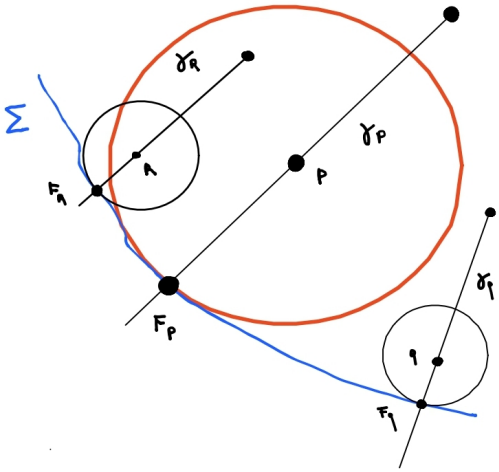
Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions



Strategy of the proof

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

It exploits the duality between viscous, distributional and variational interpretation of Laplacian bounds.

As in Caffarelli-Cordoba, the bound on the Laplacian is obtained via the viscosity theory, using comparison with the Dirichlet problem for the Laplacian.

As in Petrunin, the geometric conditions are encoded by the proof relies on planar comparison and the convexity of the distance function.

Strategy of the proof

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

It exploits the duality between viscous, distributional and variational interpretation of Laplacian bounds.

Strategy of the proof

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

Remark

It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- in [Caffarelli-Cordoba '93] the bound on the Laplacian is proved via the viscosity theory, using comparison with quadratic polynomials and the affine structure;

Strategy of the proof

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

Remark

It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- in [Caffarelli-Cordoba '93] the bound on the Laplacian is proved via the viscosity theory, using comparison with quadratic polynomials and the affine structure;
- in [Petrunin '03] quadratic polynomials are avoided but the proof relies on parallel transport and the second variation of the arc length.

Strategy of the proof

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

Remark

It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- In [Caffarelli-Cordoba '93] the bound on the Laplacian is proved via the viscosity theory, using comparison with **quadratic polynomials** and the **affine structure**;
- in [Petrunin '03] quadratic polynomials are avoided but the proof relies on **parallel transport** and the **second variation** of the arc length.

Strategy of the proof

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Our strategy is inspired by an alternative argument given in [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and sketched in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

Remark

It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- In [Caffarelli-Cordoba '93] the bound on the Laplacian is proved via the viscosity theory, using comparison with **quadratic polynomials** and the **affine structure**;
- in [Petrunin '03] quadratic polynomials are avoided but the proof relies on **parallel transport** and the **second variation** of the arc length.

- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a lower supporting function φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.

- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ satisfies with d_E being a minimizing geodesic from x to z and $\tilde{\varphi} \leq \varphi$.

- $\tilde{\varphi}$ is a convex-like function and $\Delta \tilde{\varphi} \geq c > 0$ near to x .
- The rest of the proof and lots of $\tilde{\varphi}$ apply. Geometric interpretation controls the area growth.

- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a **lower supporting function** φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.

- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \leq d_E$.
- $\tilde{\varphi}$ is a **distance-like** function and $\Delta \tilde{\varphi} > \varepsilon > 0$ near to x_Σ .
- We cut ∂E along level sets of $\tilde{\varphi}$, apply **Gauss-Green** and contradict the area minimality.

- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a **lower supporting function** φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \leq d_E$.
- $\tilde{\varphi}$ is a **distance-like** function and $\Delta \tilde{\varphi} > \varepsilon > 0$ near to x_Σ .
- We **cut** ∂E along level sets of $\tilde{\varphi}$, apply **Gauss-Green** and contradict the area minimality.

- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a **lower supporting function** φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \leq d_E$.
- $\tilde{\varphi}$ is a **distance-like** function and $\Delta \tilde{\varphi} > \varepsilon > 0$ near to x_Σ .
- We **cut** ∂E along level sets of $\tilde{\varphi}$, apply **Gauss-Green** and contradict the area minimality.

- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a **lower supporting function** φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \leq d_E$.
- $\tilde{\varphi}$ is a **distance-like** function and $\Delta \tilde{\varphi} > \varepsilon > 0$ near to x_Σ .
- We **cut** ∂E along level sets of $\tilde{\varphi}$, apply **Gauss-Green** and contradict the area minimality.

- Suppose $K = 0$ and that super-harmonicity of d_E fails. Then we find a **lower supporting function** φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- Then we consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \leq d_E$.
- $\tilde{\varphi}$ is a **distance-like** function and $\Delta \tilde{\varphi} > \varepsilon > 0$ near to x_Σ .
- We **cut** ∂E along level sets of $\tilde{\varphi}$, apply **Gauss-Green** and contradict the area minimality.

GMT and lower Ricci bounds

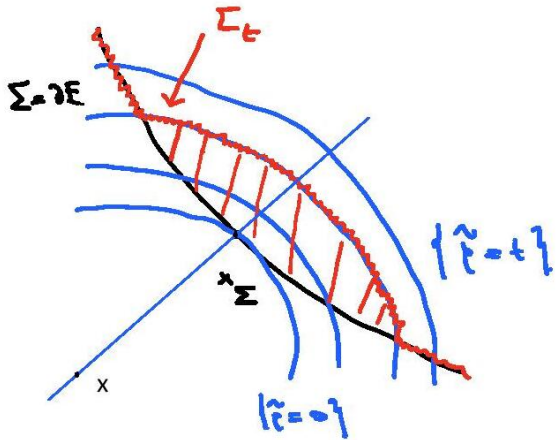
Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions



The PDE principle

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be super-harmonic. Then

$$Q_t \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{p t^{p-1}} \right\}$$

is super-harmonic for any $1 \leq p \leq \infty$ and for any $t > 0$.

- The proof completely avoids gradient estimates and requires only simple computations.

• The proof is an application of the following result which holds for the heat

$$|p \varphi_t| \leq \varphi_t^2$$

The PDE principle

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Theorem (Mondino-S. '21)

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be super-harmonic. Then

$$\mathcal{Q}_t \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{pt^{p-1}} \right\}$$

is super-harmonic for any $1 \leq p \leq \infty$ and for any $t > 0$.

- The proof completely avoids parallel transport and Jacobi fields computations.

The PDE principle

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Theorem (Mondino-S. '21)

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be super-harmonic. Then

$$\mathcal{Q}_t \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{pt^{p-1}} \right\}$$

is super-harmonic for any $1 \leq p \leq \infty$ and for any $t > 0$.

- The proof completely avoids parallel transport and Jacobi fields computations.
- It builds on the Bakry-Emery contraction estimate for the heat flow

$$|\nabla P_t f|^p \leq P_t |\nabla f|^p$$

The PDE principle

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Theorem (Mondino-S. '21)

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be super-harmonic. Then

$$\mathcal{Q}_t \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{pt^{p-1}} \right\}$$

is super-harmonic for any $1 \leq p \leq \infty$ and for any $t > 0$.

- The proof completely avoids **parallel transport** and **Jacobi fields** computations.
- It builds on the **Bakry-Émery** contraction estimate for the **heat flow**

$$|\nabla P_t f|^p \leq P_t |\nabla f|^p.$$

The PDE principle

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

Theorem (Mondino-S. '21)

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be super-harmonic. Then

$$\mathcal{Q}_t \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{pt^{p-1}} \right\}$$

is super-harmonic for any $1 \leq p \leq \infty$ and for any $t > 0$.

- The proof completely avoids **parallel transport** and **Jacobi fields** computations.
- It builds on the **Bakry-Émery** contraction estimate for the **heat flow**

$$|\nabla P_t f|^p \leq P_t |\nabla f|^p.$$

The isoperimetric problem

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

In the classical setting, area minimizing hypersurfaces have vanishing mean curvature and isoperimetric sets have constant mean curvature.

In the very abstract setting of the Ricci lower bound, what can we say?

In [Antonelli-Pasqualetto-Pozzetta-S. '21] we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) .

The isoperimetric problem

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Remark

In the classical setting, **area minimizing** hypersurfaces have *vanishing mean curvature* and **isoperimetric sets** have *constant mean curvature*.

Is there any extension of the theory for isoperimetric sets?

In [Antonelli-Pasqualetto-Pozzetta-S. '21] we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) .

The isoperimetric problem

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Remark

In the classical setting, **area minimizing** hypersurfaces have *vanishing mean curvature* and **isoperimetric sets** have *constant mean curvature*.

Is there any extension of the theory for isoperimetric sets?

In [Antonelli-Pasqualetto-Pozzetta-S. '21] we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) .

The isoperimetric problem

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Remark

In the classical setting, **area minimizing** hypersurfaces have *vanishing mean curvature* and **isoperimetric sets** have *constant mean curvature*.

Question

Is there any extension of the theory for isoperimetric sets?

In [Antonelli-Pasqualetto-Pozzetta-S. '21] we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) .

The isoperimetric problem

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Remark

In the classical setting, **area minimizing** hypersurfaces have *vanishing mean curvature* and **isoperimetric sets** have *constant mean curvature*.

Question

Is there any extension of the theory for isoperimetric sets?

In [Antonelli-Pasqualetto-Pozzetta-S. '21] we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) .

The isoperimetric problem

GMT and lower
Ricci bounds

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Remark

In the classical setting, **area minimizing** hypersurfaces have *vanishing mean curvature* and **isoperimetric sets** have *constant mean curvature*.

Question

Is there any extension of the theory for isoperimetric sets?

In **[Antonelli-Pasqualetto-Pozzetta-S. '21]** we obtain sharp Laplacian bounds for the distance function from the boundary of an isoperimetric set E inside an $\text{RCD}(K, N)$ space (X, d, \mathcal{H}^N) .

Consequences

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- As for area minimizers, the Laplacian bounds correspond to constant mean curvature and sharp information about the second variation of the area via equidistants;
- existence of isoperimetric sets is unknown on open manifolds with lower Ricci and lower volume bounds;
- by a comparison argument, the important isoperimetric result of Ambrosetti and Trombadori (1980) holds for manifolds with lower Ricci and lower volume bounds;
- we prove sharp isoperimetric properties of the non-convex regions under lower Ricci bounds that were previously known only on compact manifolds by assuming existence of a lower Ricci bound on any volume.

Consequences

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- As for area minimizers, the Laplacian bounds correspond to **constant mean curvature** and sharp information about the second variation of the area via equidistants;
- existence of isoperimetric sets is unknown on **open manifolds** with lower Ricci and lower volume bounds;
- by a **concentration-compactness** argument isoperimetric regions exist, in **generalized** sense, inside **pointed limits at infinity**;
- we prove **sharp concavity** properties of the **isoperimetric profile** under lower Ricci bounds that were previously known only on compact manifolds, or assuming existence of isoperimetric regions of any volume.

Consequences

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- As for area minimizers, the Laplacian bounds correspond to **constant mean curvature** and sharp information about the second variation of the area via equidistants;
- existence of isoperimetric sets is unknown on **open manifolds** with lower Ricci and lower volume bounds;
- by a **concentration-compactness** argument isoperimetric regions exist, in **generalized** sense, inside **pointed limits at infinity**;
- we prove **sharp concavity** properties of the **isoperimetric profile** under lower Ricci bounds that were previously known only on compact manifolds, or assuming existence of isoperimetric regions of any volume.

Consequences

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- As for area minimizers, the Laplacian bounds correspond to **constant mean curvature** and sharp information about the second variation of the area via equidistants;
- existence of isoperimetric sets is unknown on **open manifolds** with lower Ricci and lower volume bounds;
- by a **concentration-compactness** argument isoperimetric regions exist, in **generalized** sense, inside **pointed limits at infinity**;
- we prove **sharp concavity** properties of the **isoperimetric profile** under lower Ricci bounds that were previously known only on compact manifolds, or assuming existence of isoperimetric regions of any volume.

Consequences

GMT and lower Ricci bounds

Daniele Semola

Motivations

Non smooth lower Ricci bounds

Main results

Extensions

- As for area minimizers, the Laplacian bounds correspond to **constant mean curvature** and sharp information about the second variation of the area via equidistants;
- existence of isoperimetric sets is unknown on **open manifolds** with lower Ricci and lower volume bounds;
- by a **concentration-compactness** argument isoperimetric regions exist, in **generalized** sense, inside **pointed limits at infinity**;
- we prove **sharp concavity** properties of the **isoperimetric profile** under lower Ricci bounds that were previously known only on compact manifolds, or assuming existence of isoperimetric regions of any volume.

**GMT and lower
Ricci bounds**

Daniele Semola

Motivations

Non smooth
lower Ricci
bounds

Main results

Extensions

Thank you for your attention!