Research statement

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My research interests lie at the interface between Analysis and Geometry, and they are often motivated by problems involving Curvature. This concept is ubiquitous in Mathematics and Sciences more in general, a prominent example being Einstein's equations of General Relativity. Within this area, a large part of my work has been dedicated to the theory of non-smooth spaces with curvature bounds and its applications to classical problems in Geometric Analysis. A key for these applications is the observation that regularity is a highly unstable condition, a priori, and its loss can be traded for a gain of compactness. This is very much in analogy with the role of Sobolev spaces in Partial Differential Equations.

Regularity theory of non-smooth spaces with lower curvature bounds

The theory of metric spaces with upper and lower sectional curvature bounds was pioneered by Alexandrov in the Fifties, based on the previous work of Busemann and Wald. Here the word *synthetic* has to be intended in opposition to *analytic* and metric spaces are described in terms of the excess of triangles (i.e. the amount by which the sum of the angles exceeds π) with respect to the Euclidean (or spherical, or hyperbolic) behavior. The metric formulation is robust enough to pass to the singular limits that arise considering sequences of Riemannian manifolds with sectional curvature bounds. For lower curvature bounds the theory was further developed by Burago-Gromov-Perelman [13].

Developing an analogous theory in the case of lower Ricci curvature bounds turned out to be a more challenging goal, see the discussions in [18, Section 5] and [14, Appendix B]. The synthetic theory of lower Ricci bounds began with the seminal works by Sturm [31] and independently Lott-Villani [25]. The class of $\mathrm{CD}(K, N)$ metric measure spaces ("CD" standing for *Curvature-Dimension*) was introduced to provide a synthetic notion of Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $1 \leq N \leq \infty$. The treatment is based on Optimal Transport and it involves metric measure spaces, i.e. triples $(X, \mathsf{d}, \mathfrak{m})$ where (X, d) is a metric space and $\mathfrak{m}$ is a non-negative measure.
To single out spaces with a Riemannian-like behaviour from this broader class that includes finite dimensional normed vector spaces that are not Hilbert, Ambrosio-Gigli-Savaré introduced the notion of RCD space in [3] ("R" standing for Riemannian), adding the linearity of the heat flow to the CD condition. The class of $\mathrm{RCD}(K, N)$ spaces includes smooth Riemannian manifolds with Ricci curvature bounded from below and Alexandrov spaces with lower sectional curvature bounds. Moreover, it is compact with respect to (pointed) measured Gromov-Hausdorff convergence. Therefore it includes Ricci limit spaces, i.e., limits of smooth manifolds with uniform lower Ricci bounds.

Since the earliest developments of the RCD theory, one of the driving questions has been to understand the regularity of these spaces, in analogy with the more classical regularity theory in Geometric Measure Theory and for weak solutions of Partial Differential Equations.
In joint work with Bruè [12], we proved that when $N < \infty$ any $\mathrm{RCD}(K, N)$ metric measure space admits a unique well-defined *rectifiable dimension* $1 \leq n \leq N$ such that all of its tangent spaces are isometric to $\mathbb{R}^n$ away from a negligible set of points. The problem had been mentioned among the main open questions in the area in survey papers by Villani [32] and Ambrosio [1]. The result generalizes an earlier one due to Colding-Naber in the case of Ricci limit spaces [15]. However, the proof relies on a completely novel approach to the regularity of flows of Sobolev vector fields.
Another classical challenge in the theory of non-smooth spaces with lower curvature bounds is the definition and regularity of the boundary. In collaboration with Bruè and Naber [7] we proved several regularity and stability results for boundaries of $\mathrm{RCD}(K, N)$ spaces such that the rectifiable dimension n coincides with the synthetic upper bound on the dimension N. In subsequent work with Bruè and Mondino [6] the techniques from [7] are combined with a new functional inequality to relate a geometric notion of boundary to an analytic one based on the average distortion of volumes of small

balls with respect to the Euclidean one. With earlier results from [22], the main theorem of [6] leads to the positive resolution of an open question about the existence of infinite geodesics on Alexandrov spaces with empty boundary raised by Perelman and Petrunin in 1996.

Geometric measure theory in low regularity and applications

The interplay between curvature and minimal surfaces, or more general solutions of geometric variational problems involving the area functional, has been one of the cornerstones of Geometric Analysis in the last fifty years. Among the most classic examples of this deep connection are the Lévy-Gromov isoperimetric inequality for manifolds with positive Ricci curvature [19, Appendix C] and the proof by Schoen-Yau of the positive mass theorem for asymptotically flat manifolds with nonnegative scalar curvature [29]. These developments were intertwined with the birth and growth of Geometric Measure Theory in Euclidean and Riemannian setting. This celebrated connection between curvature and minimal surfaces and the relevance of the theory of singular spaces with curvature bounds motivate the attempts to develop Geometric Measure Theory in non-smooth ambient spaces with curvature bounds.

In two joint projects with Ambrosio-Bruè [2] and Bruè-Pasqualetto [11] we developed a theory of sets of finite perimeter in RCD metric measure spaces. Sets of finite perimeter are a versatile tool for studying geometric variational problems in codimension one (their boundaries should be considered as generalized hypersurfaces) and in the Euclidean case their theory was pioneered by Caccioppoli and De Giorgi in the Fifties. The state of the art of the theory in the RCD setting after [2, 11] is analogous to the Euclidean situation, to most extents. However, the extension of the classical results to this low-regularity setting required several new insights.

In more recent joint work with Mondino [27] we initiated a regularity theory for solutions of minimization problems for sets of finite perimeter over RCD spaces. The main goal is to show that the (local) minimality and the lower bound on the Ricci curvature affect the first and second variation of the area as in the case of smooth ambient spaces. Most of the usual arguments break in this setting due to the presence of singularities in the ambient space, which might be dense, and to the absence of any known regularization method. The main results of [27] answer some questions that had been raised by Gromov [20, Section 6.9]. The tools developed in [27] have been employed and generalized in [5] to answer several open questions about the isoperimetric problem on smooth complete Riemannian manifolds with Ricci curvature bounded from below.

Fundamental groups of open manifolds with nonnegative Ricci curvature

One of the main broad themes of Geometric Analysis is to understand the global implications of a given curvature condition on the topology (or even differentiable structure) of a smooth Riemannian manifold. In the case of Ricci curvature, a result of Lohkamp [24] shows that the existence of a metric with negative Ricci curvature does not restrict at all the topology in dimensions larger than 3. On the other hand, a great deal of work in the past fifty years was stimulated by a conjecture of Milnor [26] that predicted that a complete smooth Riemannian manifold with nonnegative Ricci curvature should have finitely generated fundamental group. In collaboration with Bruè and Naber we constructed a family of counterexamples to the Milnor conjecture. The conjecture was known to be true in dimensions less or equal to 3 [23] and the counterexamples were constructed first in dimension 7 in [8]. More recently in [10] we managed to lower the dimension of the counterexamples to 6 with a non-trivial extension of the techniques developed in our earlier paper.

References

[1] L. Ambrosio: *Calculus, heat flow and curvature-dimension bounds in metric measure spaces,* Proceedings of the international congress of mathematicians, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume I. Plenary lectures.

[2] L. Ambrosio, E. Bruè, **D. Semola**: *Rigidity of the* 1*-Bakry-Émery inequality and sets of finite perimeter in* RCD *spaces,* Geom. Funct. Anal. **29** (2019), no. 4, 949–1001.

[3] L. Ambrosio, N. Gigli, G. Savaré: *Metric measure spaces with Riemannian Ricci curvature bounded from below,* Duke Math. J. **163** (2014), no. 7, 1405–1490.

[4] M.-T. Anderson: *Hausdorff perturbations of Ricci-flat manifolds and the splitting theorem,* Duke Math. J. **68** (1992), no. 1, 67–82.

[5] G. Antonelli, E. Pasqualetto, M. Pozzetta, **D. Semola**: *Sharp isoperimetric comparison on non-collapsed spaces with lower Ricci bounds,* accepted for publication by Ann. Sci. Éc. Norm. Supér. (2023).

[6] E. Bruè, A. Mondino, **D. Semola**: *The metric measure boundary of spaces with Ricci curvature bounded below,* Geom. Funct. Anal. **33** (2023), no. 3, 593–636.

[7] E. Bruè, A. Naber, **D. Semola**: *Boundary regularity and stability for spaces with Ricci bounded below,* Invent. Math. **228** (2022), no. 2, 777–891.

[8] E. Bruè, A. Naber, **D. Semola**: *Fundamental Groups and the Milnor Conjecture,* preprint arXiv:2303.15347 (2023).

[9] E. Bruè, A. Naber, **D. Semola**: *Stability of tori under lower sectional curvature,* accepted for publication by Geom. Topol. (2023).

[10] E. Bruè, A. Naber, **D. Semola**: *Six dimensional counterexample to the Milnor conjecture,* preprint arXiv:2311.12155v1 (2023).

[11] E. Bruè, E. Pasqualetto, **D. Semola**: *Rectifiability of the reduced boundary for sets of finite perimeter over* $\mathrm{RCD}(K,N)$ *spaces,* J. Eur. Math. Soc. (JEMS) **25** (2023), no. 2, 413–465.

[12] E. Bruè, **D. Semola**: *Constancy of the Dimension for RCD(K,N) Spaces via Regularity of Lagrangian Flows*, Comm. Pure Appl. Math. **73** (2020), no. 6, 1141–1204.

[13] Y. Burago, M. Gromov, G. Perelman: *A. D. Aleksandrov spaces with curvatures bounded below.* Uspekhi Mat. Nauk **47** (1992), no. 2 (284), 3–51, 222.

[14] J. Cheeger, T.-H Colding: *On the structure of spaces with Ricci curvature bounded below. I.* J. Differential Geom. **46** (1997), no. 3, 406–480.

[15] T.-H. Colding, A. Naber: *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications.* Ann. Math. (2) **176** (2012), no. 2, 1173–1229.

[16] T.-H. Colding, A. Naber: *Characterization of tangent cones of noncollapsed limits with lower Ricci bounds and applications,* Geom. Funct. Anal. **23** (2013), no. 1, 134–148.

[17] J. Eells, J.-H. Sampson: *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964), 109–160.

[18] Gromov, M.: *Sign and geometric meaning of curvature,* Rend. Semin. Mat. Fis. Milano **61** (1991), 9–123.

[19] M. Gromov: *Metric structures for Riemannian and non-Riemannian spaces,* Transl. from the French by Sean Michael Bates. With appendices by M. Katz, P. Pansu, and S. Semmes. Edited by J. LaFontaine and P. Pansu. Progress in Mathematics (Boston, Mass.). 152. Boston, MA: Birkhäuser. xix, 585 p. (1999).

[20] M. Gromov: *Four Lectures on Scalar Curvature,* Perspectives in Scalar Curvature, (2023) https://doi.org/10.1142/12644.

[21] M. Gromov, R. Schoen: *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one,* Inst. Hautes Études Sci. Publ. Math. **76** (1992), 165–246.

[22] V. Kapovitch, A. Lytchak, A. Petrunin: *Metric-measure boundary and geodesic flow on Alexandrov spaces,* J. Eur. Math. Soc. (JEMS) **23** (2021), no. 1, 29–62.

[23] G. Liu: *3-manifolds with nonnegative Ricci curvature,* Invent. Math. **193** (2013), no. 2, 367–375.

[24] J. Lohkamp: *Metrics of negative Ricci curvature.* Ann. Math. (2) **140** (1994), no. 3, 655–683.

[25] J. Lott, C. Villani: *Ricci curvature for metric-measure spaces via optimal transport,* Ann. of Math. (2) **169** (2009), no. 3, 903–991.

[26] J.-W. Milnor: *A note on curvature and fundamental group,* J. Differ. Geom. **2** (1968), 1–7.

[27] A. Mondino, **D. Semola**: *Weak Laplacian bounds and minimal boundaries on non-smooth spaces with Ricci curvature lower bounds,* accepted for publication by Mem. Amer. Math. Soc., (2023).

[28] A. Mondino, **D. Semola**: *Lipschitz continuity and Bochner-Eells-Sampson inequality for harmonic maps from* $\mathrm{RCD}(K,N)$ *to* $\mathrm{CAT}(0)$ *spaces,* accepted for publication by Amer. J. Math. (2023).

[29] R. Schoen, S.-T. Yau: *On the proof of the positive mass conjecture in general relativity,* Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[30] M. Simon, P. Topping: *Local mollification of Riemannian metrics using Ricci flow, and Ricci limit spaces,* Geom. Topol. **25** (2021), no. 2, 913–948.

[31] K.-T. Sturm: *On the geometry of metric measure spaces I, II,* Acta Math. **196** (2006), no. 1, 65–131, 133–177.

[32] C. Villani: *Synthetic theory of Ricci curvature bounds,* Jpn. J. Math. (3) **11** (2016) no. 2, 219–263.