GMT and lower Ricci bounds

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Isoperimetry and lower curvature bounds

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Outline

General idea

I will discuss some recent results about the isoperimetric problem on *spaces* with lower Ricci curvature bounds.

We will see that for smooth Riemannian manifolds, the non compact case is subtler than the compact one.

It also naturally leads to study the problem on more general metric measure spaces with synthetic lower Ricci curvature bounds.

Lower Ricci bounds and GMT

$\Sigma^{N-1} \subset M^N$ is minimal and two-sided with unit normal ν. Then we can compute the second variation of the area for vector fields X such that $X = f\nu$ along Σ:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathcal{H}^{N-1}(\Phi_t(\Sigma)) = \int_\Sigma \left[|\nabla_\Sigma f|^2 - \left(|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu)\right) f^2\right] \mathrm{d}\,\mathcal{H}^{N-1} \,.$$

Theorem (Simons, *Ann. of Math.* '68)

There are no closed two sided stable minimal hypersurfaces in a closed manifold with positive Ricci curvature.

Corollary

There is no two-sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.

Isoperimetric problem and isoperimetric profile

For a smooth Riemannian manifold (M, g), with volume measure vol and codimension one surface area Per, we define the isoperimetric profile $I : [0, \mathrm{vol}(M)) \to [0, +\infty)$ by

$$I(v) := \inf \{\mathrm{Per}(\Omega) \, : \, \Omega \subset M \, , \;\; \mathrm{vol}(\Omega) = v\} \, .$$

The very same definition makes sense for metric measure spaces $(X, \mathrm{d}, \mathrm{m})$. Minimizers are called isoperimetric sets/regions.

Remark

On $\mathbb{R}^N$, endowed with the Euclidean distance and the Lebesgue measure,

$$I(v) = N\omega_N^{\frac{1}{N}} v^{\frac{N-1}{N}} \, ,$$

by the classical isoperimetric inequality.

Levy-Gromov inequality

Theorem (Gromov '86)

Let (M^N, g) be a smooth Riemannian manifold with $\mathrm{Ric} \geq N-1$. Then for any domain $\Omega \subset M$

$$\frac{\mathrm{Per}(\Omega)}{\mathrm{vol}(M)} \geq \frac{\mathrm{Per}(\Omega^*)}{\mathrm{vol}(\mathbb{S}^N)},$$

where $\Omega^* \subset \mathbb{S}^N$ is a ball such that

$$\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(M)} = \frac{\mathrm{vol}(\Omega^*)}{\mathrm{vol}(\mathbb{S}^N)}.$$

Remark

In the original proof, the infinitesimal estimate obtained by the second variation formula is globalized to the whole manifold.

Nonnegative Ricci curvature and Euclidean volume growth

By Bishop-Gromov, if (M^N, g) has nonnegative Ricci curvature, then the limit

$$\lim_{R\to\infty} \frac{\mathrm{vol}(B_R(p))}{\omega_N R^N} \in [0,1]$$

exists and it is independent of the base point p. We shall call it AVR, standing for Asymptotic Volume Ratio.

Theorem (See next slide)

Let (M^N, g) be complete with $\mathrm{Ric} \geq 0$. Then

$$\mathrm{Per}(E) \geq N\omega_N^{\frac{1}{N}} \mathrm{AVR}^{\frac{1}{N}} \left(\mathrm{vol}(E)\right)^{\frac{N-1}{N}} ,$$

for any Borel set $E \subset M$.

Approaches

- [Agostiniani-Fogagnolo-Mazzieri, *Invent. Math.* '20] $N = 3$, Geometric Flows;
- [Brendle, *CPAM* '20], Optimal Transport;
- [Fogagnolo-Mazzieri, *JFA* '21], $N \leq 7$, Geometric Flows;
- [Balogh-Kristály, *Math. Ann.* '21], Brunn-Minkowski;
- [Antonelli-Pasqualetto-Pozzetta-S., '22];
- [Cavalletti-Manini, '22], Localization technique.

Remark

The last three approaches cover more general ambient spaces. The most general one is in [Balogh-Kristály '21].

Differential inequalities for the isop. profile, I

On model spaces with constant sectional curvature $K/(N-1) \in \mathbb{R}$ and dimension $N \geq 2$ the isoperimetric profile $I_{K,N}$ satisfies

$$-I''_{K,N} I_{K,N} = K + \frac{\left(I'_{K,N}\right)^2}{N-1} .$$

By [Bayle, *IMRN* '04] (see also [Bavard-Pansu, *ASENS* '86], [Morgan-Johnson, *IUMJ* '00] and [Ni-Wang, *JGA* '16]):

Theorem

Let (M^N, g) be a closed smooth Riemannian manifold with $\mathrm{Ric} \geq K$. Then

$$-I'' I \geq K + \frac{(I')^2}{N-1}$$

in the sense of distributions on $(0, \mathrm{vol}(V))$.

The non compact case

Theorem (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let (M^N, g) be a complete smooth Riemannian manifold with $\mathrm{Ric} \geq K$. Then

$$-I'' I \geq K + \frac{(I')^2}{N-1}$$

in the sense of distributions on $(0, \mathrm{vol}(V))$.

The statement was previously known:

- for $N = 2$ and $K = 0$, by [Ritoré, *JGA* '02];
- under uniform bounded geometry assumptions, by [Mondino-Nardulli, *CAG* '16].

Consequences

Corollary

Let (M^N, g) *be a smooth Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Then the function*

$$v \mapsto \frac{I(v)}{v^{\frac{N-1}{N}}}$$

is monotone decreasing.

Corollary (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let (M^N, g) *be a smooth Riemannian manifold with* $\mathrm{Ric} \geq K$. *Then*

$$\lim_{v\to 0} \frac{(I(v))^N}{v^{N-1}} = N^N \omega_N \lim_{r\to 0} \left(\inf_{p\in M} \frac{\mathrm{vol}(B_r(p))}{v_{K/(N-1),N}(r)} \right) ,$$

where $v_{K/(N-1),N}(r)$ *is the volume of the ball of radius* r *in the model space of dimension* N *and sectional curvature* $K/(N-1)$.

Generalized existence, I

If we drop the compactness, the direct method of calculus of variations fails to guarantee existence of isoperimetric regions.

Remark

The loss of compactness can be handled with the more sophisticated concentration-compactness method, first introduced in [Lions, *AIHP* '84].

[Ritoré-Rosales, *TAMS* '04]: for a minimizing sequence for volume v, part of the mass converges to an isoperimetric region with volume $\leq v$, the remaining part diverges to infinity.

[Nardulli, *Asian J. Math* '14]: if the geometry at infinity is "uniformly bounded", the escaping parts converge to isoperimetric regions in some pointed limits at infinity, that are smooth Riemannian manifolds too.

Compactness and limits under lower Ricci bounds

Theorem (Gromov '82)

The class $\mathcal{M}_{N,K}$ of smooth Riemannian manifolds with $\dim \le N$ and $\mathrm{Ric} \ge K$ is precompact w.r.t. the pointed Gromov-Hausdorff topology.

Remark

Understanding of $\mathcal{M}_{N,K}$ is tightly linked with understanding of its closure with respect to the Gromov-Hausdorff topology.

- Several contributions to the theory of Ricci limit spaces by Fukaya, Anderson, Cheeger, Colding, Tian, Naber, Kapovitch, Wilking, Jiang, etc. .
- Deep influence over related areas such as Kähler geometry: [Chen-Donaldson-Sun, *J. Amer. Math. Soc.* '15], [Tian, *Comm. Pure Appl. Math.* '15], [Liu, *Ann. of Math.* '16], etc. .

Generalized existence, II

Theorem (Antonelli-Nardulli-Pozzetta, *ESAIM: COCV* '22)

Given a complete Riemannian manifold (M^N, g) with $\mathrm{Ric} \geq K$ and $\mathrm{vol}(B_1(x)) > v_0 > 0$ for every $x \in X$, for any $v > 0$ there exist:

- *a finite collection of $M \in \mathbb{N}$ Ricci limit spaces $(X_i, \mathrm{d}_i, \mathcal{H}^N) = \lim_{k\to\infty}(M, \mathrm{d}_g, \mathcal{H}^N, p_i^k)$ in the measured pGH topology, where $\mathrm{d}_g(p_i^k, p) \to \infty$ as $k \to \infty$;*
- *isop. regions $\Omega_0, \Omega_1, \ldots, \Omega_i$ with $\Omega_0 \subset M$, $\Omega_i \subset X_i$ such that:*
  - i) *there is no loss of mass:*
  $$\sum_{i=0}^{M} \mathcal{H}^N(\Omega_i) = v\,;$$
  - ii) *the value of the isoperimetric profile (of M) is achieved:*
  $$\sum_{i=0}^{M} \mathrm{Per}_i(\Omega_i) = I(v)\,.$$

Questions

Question

Do area minimizing hypersurfaces in non smooth spaces with lower curvature bounds have vanishing mean curvature?
Are isoperimetric surfaces CMC?

Question

Does a lower Ricci curvature bound affect the second variation of the area in non smooth ambient spaces?

Motivations

- Understand Curvature, (cf. with [Gromov '19]);
- GMT on singular spaces as a new tool for classical GMT.

Remarks

Remark

It seems worthwhile to develop a theory independent of the existence of smooth approximating sequences.

Cf. with the question of non branching for geodesics in Ricci limit spaces [Colding-Naber, *Ann. of Math.* '12], [Deng '20].

Remark

Smoothness gets lost, but the lower Ricci curvature bound is stable, in suitable sense.

RCD spaces

$\mathrm{RCD}(K,N)$ metric measure spaces (X,d,m) are "Riemannian" spaces with Ricci bounded from below by $K\in\mathbb{R}$, dimension bounded above by $1\le N<\infty$.

Recall the Bochner identity:

$$\frac{1}{2}\Delta|\nabla u|^2=||\mathrm{Hess}\,u||^2_{\mathrm{HS}}+\nabla u\cdot\nabla\Delta u+\mathrm{Ric}(\nabla u,\nabla u)\,.$$

Definition

A m.m.s. (X,d,m) is $\mathrm{RCD}(K,N)$ if:

- $W^{1,2}$ is a Hilbert space and functions with bounded gradient are Lipschitz;
- for sufficiently many test functions $u:X\to\mathbb{R}$,

$$\frac{1}{2}\Delta|\nabla u|^2\ge\frac{(\Delta u)^2}{N}+\nabla u\cdot\nabla\Delta u+K|\nabla u|^2\,.$$

Remarks

Contributions by several authors after [Sturm, *Acta Math.* '06], [Lott-Villani, *Ann. of Math.* '09], [Ambrosio-Gigli-Savaré, *Duke Math. J.* '14].

Examples of $\mathrm{RCD}(K, N)$ spaces are:

- Ricci limit spaces;
- Alexandrov spaces with curvature bounded below [Petrunin '09], [Zhang-Zhu '09];
- cones and spherical suspensions [Ketterer '13, '15];
- quotients of smooth Riemannian manifolds with lower Ricci bounds w.r.t isometric group actions [Galaz-García-Kell-Mondino-Sosa, '18].

Regularity of non collapsed RCD spaces

Consider an $\mathrm{RCD}(K, N)$ space $(X, \mathrm{d}, \mathcal{H}^N)$. Here $\mathrm{m} = \mathcal{H}^N$. We assume that it has no boundary for simplicity. Then:

- (X, d) is N-rectifiable [Mondino-Naber, *JEMS* '14] after [Cheeger-Colding, *JDG* '97];
- (X, d) is bi-Hölder homeomorphic to a smooth manifold away from a set of codimension two [Kapovitch-Mondino, *Geom. Topol.* '19] after [Cheeger-Colding, *JDG* '97];
- all tangent cones are metric cones and there is a stratification of the singular set [De Philippis-Gigli, *J. Éc. polytech. Math.* '18] after [Cheeger-Colding, *JDG* '97].

Bad news

- The set of singular points (i.e., with non Euclidean tangent cones) might be dense [Otsu-Shioya, *JDG* '94];
- [Colding-Naber, *Adv. Math.* '13] build examples with no singular points where the distance is not induced by C^α Riemannian metrics for any $\alpha > 0$;
- conjecturally ([Yau '90], [Naber '20], ...), the scalar curvature is a measure. There are elementary examples where it is singular (with respect to the volume);

Area minimizing hypersurfaces

We deal with sets of finite perimeter.

- Euclidean theory pioneered by Caccioppoli and De Giorgi;
- for $E \subset \mathbb{R}^N$ smooth, $\mathrm{Per}(E, \cdot) = \mathcal{H}^{N-1} \llcorner \partial E$;
- on metric measure spaces from [Ambrosio, *Adv. Math.* '02];
- on $\mathrm{RCD}(K, N)$ spaces well understood (perimeter equals codimension one measure, rectifiability with well defined dimension, Gauss-Green formula) after [Ambrosio-Bruè-S., *GAFA* '18], [Bruè-Pasqualetto-S., *JEMS* '19].

Definition

We say that $E \subset X$ is a local perimeter minimizer if for any $x \in E$ there is a neighbourhood $U_x \ni x$ such that

$$\mathrm{Per}(E, U_x) \leq \mathrm{Per}(F, U_x)\,, \quad \text{if } F \Delta E \Subset U_x\,.$$

Laplacian comparison for minimizers

For $K \in \mathbb{R}$ and $1 \le N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)}\,\tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)}\,\tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Theorem (Mondino-S. '21)

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a local perimeter minimizer. Let $\mathrm{d}_{\overline{E}} : X \setminus \overline{E} \to [0, \infty)$ be the distance function from $\overline{E}$. Then

$$\Delta \mathrm{d}_{\overline{E}} \le \tau_{K,N} \circ \mathrm{d}_{\overline{E}} \quad \text{on } X \setminus \overline{E}\,.$$

The analogous statement holds for $\mathrm{d}_{X\setminus E}$ on E.

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Remarks

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $\mathrm{RCD}(K, N)$ spaces. They can be understood distributionally;
- the bounds are sharp and attained on the model spaces;
- on $\mathbb{R}^n$ the bounds imply that ∂E is a viscosity solution of the minimal surfaces equation [Savin, *Comm. PDEs* '07].
- no need to talk about mean curvature of the area minimizing boundary;
- in smooth ambient spaces, the bounds follow from [Gromov '82] (see also [Wu, *Acta Math.* '79]). The argument does not carry over in the present setting.

Further remarks

Remark

The argument for the Laplacian comparison is very flexible. It works for general solutions of minimization problems (bubbles, isoperimetric sets, ...).

Applying the Gauss-Green theorem on a tubular neighbourhood of the boundary we get sharp estimates for the first and second variation of the area.

Remark

In combination with the generalized existence of isoperimetric regions this is the main tool to prove the isoperimetric comparison estimates.

Open questions, I

Open question

Can we obtain every function $I : (0, \infty) \to (0, \infty)$ such that $I^{\frac{N}{N-1}}$ is concave as the isoperimetric profile of a complete N-dim. manifold with nonnegative Ricci curvature?

Open question

Let (M^N, g) be a smooth Riemannian manifold with $\mathrm{Ric} \geq 0$ and $\mathrm{AVR} > 0$. Is it true that isoperimetric regions exist in M for any sufficiently large volume?

- By [Antonelli-Bruè-Fogagnolo-Pozzetta, *Calc. Var.* '22] the answer is affirmative if $\mathrm{Sec} \geq 0$.
- The answer seems not clear even for Ricci flat manifolds.

Open questions, II

Conjecture

Boundaries of isoperimetric sets in $\mathrm{RCD}(K,N)$ spaces $(X,\mathrm{d},\mathcal{H}^N)$ are homeomorphic to topological manifolds away from sets of ambient Hausdorff codimension four.

Remark

The spherical suspension over $\mathbb{RP}^2$ with standard metric has two non manifold points. Taking the cone over it we could check the sharpness of the above conjecture.

Thank you for your attention!