A quantitative Obata theorem via localization[1]

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[1] Based on a joint work with Fabio Cavalletti (SISSA) and Andrea Mondino (Oxford).

Plan

The classical problem

Finding a lower bound for the smallest eigenvalue of $-\Delta$ under geometric assumptions.

Deep connection between the geometry of a space and spectral properties of the Laplacian is at the core of geometric analysis.

First eigenvalue of the Laplacian: a lower bound of the first eigenvalue gives an upper bound of the constant in the Poincaré inequality.

- **Domains in the Euclidean space**: Lord Rayleigh, Polya-Szego, Faber-Krahn, Payne, Weinberger, ...
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Variational characterization of the first eigenvalue

For a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ one can define the non-negative real number $\lambda^{1,2}_{(X,\mathrm{d},\mathfrak{m})}$ as follows

$$\lambda^{1,2}_{(X,\mathrm{d},\mathfrak{m})} := \inf\left\{ \frac{\int_X |\nabla u|^2\,\mathfrak{m}}{\int_X |u|^2\,\mathfrak{m}} : u \in \mathrm{Lip}(X) \cap L^2(X,\mathfrak{m}),\ u \neq 0,\ \int_X u\,\mathfrak{m} = 0 \right\}, \quad (1)$$

where $|\nabla u|$ is the slope (also called local Lipschitz constant) of the Lipschitz function u given by

$$|\nabla u|(x) = \limsup_{y\to x} \frac{|u(x)-u(y)|}{\mathrm{d}(x,y)} \quad \text{if } x \text{ is not isolated, 0 otherwise.}$$

Remarks:

i) (1) is consistent with the smooth case and incorporates Neumann conditions;

ii) thanks to (1) we can talk about the first eigenvalue of the Laplacian on any m.m.s..

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Setting: N-dimensional Riemannian manifolds with Ricci curvature bounded below by $N-1$.

Theorem (Lichnerowicz '58)

Let (M, g) be an N-dimensional Riemannian manifold with Ricci bounded below by $N-1$. Then $\lambda^{1,2}(M) \geq N$

Question: what about the equality cases?

Theorem (Obata '62)

Let (M, g) be an N-dimensional Riemannian manifold with Ricci curvature bounded below by $N-1$. If the first eigenvalue of the Laplacian on (M, g) is N, then the Riemannian manifold is isometric to S^N. Moreover the eigenfunctions associated to the first eigenvalue are of the form $\cos(d(p, \cdot))$ for some $p \in S^N$.

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Remarks and key ideas

For a smooth Riemannian manifold (M^N, g) with Ricci curvature bounded below by $N - 1$ it holds:

- $\lambda^{1,2}(M) \geq N$ (Lichnerowicz '58);
- $\mathrm{diam}(M) \leq \pi$ (Myers '41).

If equality holds in one of the items above, then it holds in the other and (M^N, g) is the canonical sphere (Cheng '75), (Obata '62).

Key tool: reduction to one dimensional analysis via Jacobi fields computations.

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Quantitative inequalities

- **Starting point:** geometric/functional inequality: minimize a functional F under suitable constraints;
- Intermediate step: study of the rigidity cases: when is the minimum attained?
- Expectation: configurations where minimality is "almost attained" are "almost rigid";
- **Quantitative inequalities:** find a distance d over the space of configurations and $f : [0, +\infty) \to [0, +\infty)$ decreasing and with $f(x) \downarrow 0$ as $x \downarrow 0$ such that, if

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$$d(u, v) \le f(F(v) - \min F)$$

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The existing literature

Quantitative results about the diameter obtained in (Croke '82) and (Bérard-Besson-Gallot '85):

$$\pi - \mathrm{diam}(M) \leq C(N) \left(\lambda^{1,2}(M) - N\right)^{1/N}.$$

Results about the shape of eigenfunctions obtained in (Petersen '99) and (Bertrand '07):

Theorem (Petersen '99)

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Quantitative results about the diameter obtained in (Croke '82) and (Bérard-Besson-Gallot '85):

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Motivations

Remark

To argue by compactness you need to work where there is compactness: non smooth geometry (Ricci limits) enters into play.

Enlarging the study to $\mathrm{RCD}(N-1,N)$ spaces it is possible to characterize rigidity:

Theorem (Ketterer '15)

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(N-1,N)$ m.m.s. such that $\lambda^{1,2}(X,\mathsf{d},\mathfrak{m})=N$. Then $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to a spherical suspension over an $\mathrm{RCD}(N-2,N-1)$ m.m.s. $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$.
Moreover, any eigenfunction associated to the first eigenvalue N is a multiple of $\cos(\mathsf{d}_P)$, where P is a pole in the spherical suspension structure.

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For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\mathrm{RCD}(N-1, N)$ m.m.s. $(X, \mathrm{d}, \mathrm{m})$ and for any $u \in W^{1,2}(X, \mathrm{d}, \mathrm{m})$ with $\|u\|_{L^2(\mathrm{m})} = 1$, $\int_X u \, \mathrm{dm} = 0$ and

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The statement above is not quantitative.
The statement in (Petersen '99) can be made quantitative but requires strong PDE tools and the approach is very much relying on linearity.

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The framework

The curvature-dimension condition $\mathrm{CD}(K, N)$ introduced by (Sturm '06) and (Lott-Villani '09) gives a way to talk about dimension upper bounds coupled with lower Ricci curvature bounds for metric measure spaces.

The essentially non branching condition was introduced in (Rajala-Sturm '14). A m.m.s. is e.n.b. if any optimal geodesic plan between probabilities a.c. with respect to the reference measure is concentrated on non branching geodesics.

We work in the framework of essentially non branching $\mathrm{CD}(N-1, N)$ metric measure spaces. This framework includes:

- weighted Riemannian manifolds;
- Ricci limits;
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Main results

Theorem (Cavalletti-Mondino-S.'19)

There exists $C = C_N > 0$ such that, if $(X, \mathrm{d}, \mathfrak{m})$ is an essentially non branching $\mathrm{CD}(N-1, N)$ m.m.s., then

$$C_N(\pi - diam(X))^N \leq \lambda^{1,2}_{(X,\mathrm{d},\mathfrak{m})} - N.$$

Theorem (Cavalletti-Mondino-S.'19)

For any $N \in (1, \infty)$ there exist $C(N) > 0$ and $\delta_0 = \delta_0(N) > 0$ such that if $(X, \mathrm{d}, \mathfrak{m})$ is an essentially non branching $\mathrm{CD}(N-1, N)$ m.m.s. and $u \in \mathrm{Lip}(X)$ is such that $\int_X u \,\mathrm{d}\mathfrak{m} = 0$, $\int_X u^2 \,\mathrm{d}\mathfrak{m} = 1$ and

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then there exists a distinguished point $P \in X$ such that

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The tool: localization

Theorem (Cavalletti-Mondino '17)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. $\mathrm{CD}(N-1, N)$ m.m.s. for some $N \in (1, \infty)$. Let $f : X \to \mathbb{R}$ be such that $\int_X f\, \mathfrak{m} = 0$. Then X can be written as the disjoint union of Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q \in Q}$ and a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}}$, $\{\mathfrak{m}_q\}_{q\in Q}$ such that:

- *For any $\mathfrak{m}$-measurable set $B \subset \mathcal{T}$ it holds*

$$\mathfrak{m}(B) = \int_Q \mathfrak{m}_q(B)\, \alpha(dq),$$

  where α is a probability measure over Q defined on the quotient σ-algebra $\mathcal{Q}$.

- *For α-almost every $q \in Q$, the set X_q is a geodesic and $\mathfrak{m}_q$ is supported on it. Moreover $q \mapsto \mathfrak{m}_q$ is a $\mathrm{CD}(N-1, N)$ disintegration.*

- *For α-a.e. $q \in Q$, $\int_{X_q} f\, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$-a.e. in Z.*

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Theorem (Cavalletti-Mondino '17)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an e.n.b. $\mathrm{CD}(N-1, N)$ m.m.s. for some $N \in (1, \infty)$. Let $f : X \to \mathbb{R}$ be such that $\int_X f\, \mathfrak{m} = 0$. Then X can be written as the disjoint union of Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q \in Q}$ and a disintegration of $\mathfrak{m} \llcorner_{\mathcal{T}}$, $\{\mathfrak{m}_q\}_{q \in Q}$ such that:

- *For any $\mathfrak{m}$-measurable set $B \subset \mathcal{T}$ it holds:*

$$\mathfrak{m}(B) = \int_Q \mathfrak{m}_q(B)\, \alpha(dq),$$

  where α is a probability measure over Q defined on the quotient σ-algebra $\mathcal{Q}$.

- *For α-almost every $q \in Q$, the set X_q is a geodesic and $\mathfrak{m}_q$ is supported on it. Moreover $q \mapsto \mathfrak{m}_q$ is a $\mathrm{CD}(N-1, N)$ disintegration.*

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Remarks about the localization tool

Remark

Curvature-dimension bounds involve the coupling of distance and measure.

On weighted manifolds $(M^n, \mathsf{d}_g, e^{-V}\mathrm{vol}_g)$ the modified Ricci tensor

$$\mathrm{Ric}_{N,n} = \mathrm{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N-n}$$

is the relevant object to look at for the $\mathrm{CD}(K,N)$ condition.

Localization simplifies as much as possible the metric space. The relevant information is encoded into the measure (Cavalletti-Mondino '17), (Klartag '17) after (Payne-Weinberger '60), (Gromov-Milman '87),... .

Weighted intervals $(I, \mathsf{d}_{eucl}, h\mathcal{L}^1)$ are $\mathrm{CD}(N-1,N)$ spaces if and only if

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Strategy of the proof: step 1

Apply the localization tool to the function u and get the corresponding disintegration.

Set $c_q^2 = \int_{X_q} u^2 \mathrm{d}\mathfrak{m}_q$.

$$\begin{aligned}
\delta(u) = \int_X |\nabla u|^2 \mathrm{d}\mathfrak{m} - N &\geq \int_Q \left(\int_{X_q} \frac{|u_q'|^2}{c_q^2} \mathrm{d}\mathfrak{m}_q \right) c_q^2 \, \mathrm{d}\alpha - N \\
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Smallness of the deficit δ implies smallness of the 1d deficits $\delta(u_q)$ on average.

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One dimensional results

The starting point arguing via localization is a study of the problem on $\mathrm{CD}(N-1,N)$ weighted intervals.

Theorem

There exists a constant $C_N > 0$ such that, if $(I, \mathsf{d}_{eucl}, \mathfrak{m})$ is a one dimensional $\mathrm{CD}(N-1,N)$ m.m.s., then

$$C_N(\pi - diam(I))^N \le \lambda^{1,2}_{(I,\mathsf{d}_{eucl},\mathfrak{m})} - N.$$

Theorem (Cavalletti-Mondino-S. '19)

For every $N > 1$ there exist constants $C = C(N) > 0$ and $\delta_0 = \delta_0(N) > 0$ such that, if $([0,D], \mathsf{d}_{eucl}, \mathfrak{m})$ is a $\mathrm{CD}(N-1,N)$ m.m.s. and $u \in \mathrm{Lip}(I)$ satisfies $\int u \mathrm{d}\mathfrak{m} = 0$, $\int u^2 \mathrm{d}\mathfrak{m} = 1$ and $\lambda := \int |u'|^2 \mathrm{d}\mathfrak{m} - N \le \delta_0$, then

$$\min\left\{ \left\| u - \sqrt{N+1}\cos(\cdot)\right\|_{L^2(\mathfrak{m})}, \left\| u + \sqrt{N+1}\cos(\cdot)\right\|_{L^2(\mathfrak{m})} \right\} \le C\delta^{\min\{1/2,1/(N)\}}.$$

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We fix $0 < \beta < 1$ and let Q_ℓ be the set of "long rays" q for which $\delta(u_q) \leq \delta^\beta$.
By Chebyschev there are many long rays (w.r.t. $c_q^2\alpha(\mathrm{d}q)$).

Theorem (1)

There exist $C(N) > 0$ and $\alpha(N) > 0$ such that

$$\int_{Q_\ell} \left| c_q - \frac{1}{\alpha(Q_\ell)} \int_{Q_\ell} c_q \,\mathrm{d}\alpha(q) \right|^2 \mathrm{d}\alpha(q) \leq C(N)\delta^{\alpha(N)}.$$

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There exist $C(N) > 0$ and $\beta(N) > 0$ such that

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$$\int_{Q_\ell} \left| c_q - \frac{1}{\alpha(Q_\ell)} \int_{Q_\ell} c_q \, \mathrm{d}\alpha(q) \right|^2 \mathrm{d}\alpha(q) \le C(N) \delta^{\alpha(N)}.$$

Theorem (2)

There exist $C(N) > 0$ and $\beta(N) > 0$ such that

$$\alpha(Q_\ell) \ge 1 - C(N)\delta^{\beta(N)}.$$

Strategy of the proof: step 2

We fix $0 < \beta < 1$ and let Q_ℓ be the set of "long rays" q for which $\delta(u_q) \leq \delta^\beta$.
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Step 2 is the most technical part of the proof. It requires new ideas with respect to (Cavalletti-Maggi-Mondino '17).

Isoperimetric inequality: perimeter is minimized at volume fixed.

Spectral gap inequality: energy is minimized with mean and L^2-norm fixed. Mean is fixed by localization, it is necessary to control the L^2-norms of the restrictions to rays.

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About the proof

The heuristic above presents two main difficulties:

- seems to rely on infinitesimal hilbertianity;
- seems to rely on *good* properties of the quotient Q.

The proofs of Theorem 1 and Theorem 2 are obtained applying the Poincaré inequality on small balls centred at suitably chosen *poles*.

- we estimate the variance of u on the ball in terms of the variance of $q \mapsto c_q$ and of the measure of long rays;
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a) By Step 1 u is "concentrated" on long rays. In particular, there is a ray with small deficit: we get the diameter estimate;

b) by Theorem 1 c_q is "almost constant" on Q_ℓ;

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Remarks and open problems

Main improvements with respect to the literature on the topic:

- explicit behaviour in the quantitative inequality;
- extension to the non smooth framework;
- approach relying on the variational characterization rather than on PDE tools.

Open problems:

- sharpness of the exponent;
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