

A quantitative Obata theorem via localization¹

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¹Based on a joint work with Fabio Cavalletti (SISSA) and Andrea Mondino (Oxford).

Plan

- 1 Spectral gap: an introduction
- 2 Introduction to the main results
- 3 Quantitative Obata via localization
- 4 Comments and remarks

The classical problem

Finding a lower bound for the smallest eigenvalue of $-\Delta$ under geometric assumptions.

Deep connection between the **geometry** of a space and **spectral** properties of the **Laplacian** is at the core of geometric analysis.

First eigenvalue of the Laplacian: a lower bound of the **first eigenvalue** gives an upper bound of the constant in the **Poincaré inequality**.

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- Domains in the Euclidean space: Lord Rayleigh, Polya-Szego, Faber-Krahn, Payne, Weinstberger, ...
- Curved spaces: Obata, Li, Yang, ...

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Variational characterization of the first eigenvalue

For a m.m.s. (X, d, m) one can define the non-negative real number $\lambda_{(X,d,m)}^{1,2}$ as follows

$$\lambda_{(X,d,m)}^{1,2} := \inf \left\{ \frac{\int_X |\nabla u|^2 m}{\int_X |u|^2 m} : u \in \text{Lip}(X) \cap L^2(X, m), u \neq 0, \int_X u m = 0 \right\}, \quad (1)$$

where $|\nabla u|$ is the slope (also called local Lipschitz constant) of the Lipschitz function u given by

$$|\nabla u|(x) = \limsup_{y \rightarrow x} \frac{|u(x) - u(y)|}{d(x, y)} \quad \text{if } x \text{ is not isolated, } 0 \text{ otherwise.}$$

Remarks:

- (1) is consistent with the smooth case and incorporates Neumann conditions;
- Remark 1.11: we can talk about the first eigenvalue of the Laplacian on any metric measure space.

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Remarks:

- 1) $\lambda_{(X,d,m)}^{1,2}$ is consistent with the spectral gap and isoperimetric constant of (X, d, m) .
- 2) $\lambda_{(X,d,m)}^{1,2} = 0$ if and only if (X, d, m) is bipartite.
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Lichnerowicz and Obata theorems

Setting: N -dimensional Riemannian manifolds with Ricci curvature bounded below by $N - 1$.

Theorem (Lichnerowicz '58)

Let (M, g) be an N -dimensional Riemannian manifold with Ricci bounded below by $N - 1$. Then $\lambda^{1,2}(M) \geq N$.

Question: what about the equality cases?

Theorem (Obata '62)

Let (M, g) be an N -dimensional Riemannian manifold with Ricci curvature bounded below by $N - 1$. If the first eigenvalue of the Laplacian on (M, g) is N , then the Riemannian manifold is isometric to S^N . Moreover the eigenfunctions associated to the first eigenvalue are of the form $\cos(d(p, \cdot))$ for some $p \in S^N$.

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Answer: If $\lambda^{1,2}(M) = N$, then (M, g) is isometric to a sphere S^N . Moreover, the eigenfunctions associated to the first eigenvalue are of the form $\cos(\rho(x, p))$ for some $p \in S^N$.

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Remarks and key ideas

For a smooth Riemannian manifold (M^N, g) with Ricci curvature bounded below by $N - 1$ it holds:

- $\lambda^{1,2}(M) \geq N$ (Lichnerowicz '58);

- $\text{diam}(M) \leq \pi$ (Myers '36).

If equality holds in one of the items above, then it holds in the other and (M^N, g) is the canonical sphere (Cheng '75), (Obata '62).

Key tool: reduction to one dimensional analysis via Jacobi fields computations.

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For a smooth Riemannian manifold (M^N, g) with Ricci curvature bounded below by $N - 1$ it holds:

- $\chi^2(M) \geq N$ (Lichnerowicz '58);
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Quantitative inequalities

- Starting point: geometric/functional inequality: minimize a functional F under suitable constraints;
- important next step: study of the rigidity cases: when is the minimum attained?
- Existence of configurations where rigidity is "almost achieved" and "almost optimal";
- Quantitative inequalities that a) determine the rigidity cases of configurations $\{x_i\} \in \mathbb{R}^d, \{r_i\} \rightarrow \{R_i, +\infty\}$ decreasing and with $\sum r_i \leq 1$ (up to some factor).

$$\min_{\mathbb{R}^d} F = F(\alpha),$$

Then

$$d(\alpha, \alpha') \leq C(F(\alpha) - \min F)$$

Quantitative inequalities

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→ important step: study of the equality case, when is the minimum attained?

→ Example: Gagliardo-Nirenberg inequality: sharp constant and extremal functions

→ Example: Poincaré inequality: sharp constant for functions vanishing on a part of the boundary

→ Example: Sobolev inequality: sharp constant and extremal functions

→ Example: isoperimetric inequality: sharp constant and extremal sets

→ Example: Brunn-Minkowski inequality: sharp constant and extremal sets

Quantitative inequalities

- **Starting point:** geometric/functional inequality: minimize a functional F under suitable constraints;
- Intermediate step: study of the rigidity cases: when is the minimum attained?
- Describe the configurations where minimum is "almost attained" and quantify this.
- Quantitative version of the inequality: how a distance from the set of configurations where $F(\Omega, \mu) \rightarrow \inf$ is related with $|F(\Omega, \mu) - \inf|$ such that $F(\Omega, \mu) \geq \inf F + c|\Omega|^\alpha$.
- Example: $\mu = \chi_\Omega$, $F(\Omega, \mu) = \text{Per}(\Omega)$ and $c|\Omega|^\alpha = c|\Omega|^{-1}$ (isoperimetric inequality):
$$\text{Per}(\Omega) \geq c|\Omega|^{-1} \Rightarrow \text{Per}(\Omega) \leq c|\Omega|^{-1} + \inf F$$

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• **Examples:** \mathcal{C}^1 functions on the sphere S^2 (Poincaré inequality, Sobolev inequality, isoperimetric inequality)

• **Examples:** \mathcal{C}^1 functions on the ball B^2 (Poincaré inequality, Sobolev inequality, isoperimetric inequality)

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● **Quantitative Inequalities:** find a distance d over the space of configurations such that $(\mathcal{R}_\epsilon \rightarrow \mathcal{R}_0)$ decreasing and with $(\mathcal{R}_\epsilon) \subset \mathcal{R}_0$

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$$\min_{v \in X} F = F(u),$$

then

$$d(u, v) \leq f(F(v) - \min F).$$

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Examples

- Isoperimetric inequalities in the Euclidean space: (Fusco-Maggi-Pratelli '08), (Figalli-Maggi-Pratelli '10), (Cicalese-Leonardi '12);
- spectral gap inequalities for domains in the Euclidean space: (Fusco-Maggi-Pratelli '09), (Brasco-Pratelli '12), (Brasco-De Philippis-Velichkov '15);
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The existing literature

Quantitative results about the diameter obtained in (Croke '82) and (Bérard-Besson-Gallot '85):

$$\pi - \text{diam}(M) \leq C(N) (\lambda^{1,2}(M) - N)^{1/N}.$$

Results about the shape of eigenfunctions obtained in (Petersen '99) and (Bertrand '07):

Quantitative shape of eigenfunctions

There exists an increasing function $r: [0, +\infty) \rightarrow [0, +\infty)$ with $r(0) = 0$, $r(x) \rightarrow 0$ as $x \rightarrow 0$ and such that, for any N -dimensional Riemannian manifold (M, g) with first nonzero eigenvalue λ , there exists $\rho \in \mathbb{R}^+$ such that the following holds. If u is an eigenfunction of the Laplacian associated to the eigenvalue λ and with $\|u\|_2^2 = 1/(N+1)$, then

$$\|u - \cos(d_{g,x}(\cdot))\|_{\infty} \leq r(\lambda - N).$$

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$$\pi - \text{diam}(M) \leq C(N) (\lambda^{1,2}(M) - N)^{1/N}.$$

Results about the shape of eigenfunctions obtained in (Petersen '99) and (Bertrand '07):

There exists an increasing function $r: [0, +\infty) \rightarrow [0, +\infty)$ with $r(0) = 0$, $r(1) = \frac{\pi}{2}$ and such that for any N -dimensional Riemannian manifold (M, g) with first nonzero eigenvalue λ , there exists $\rho \in \mathbb{R}^+$ for which the following holds: if u is an eigenfunction of the Laplacian associated to the eigenvalue λ and with $\|u\|_2^2 = 1/(N+1)$, then

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Motivations

Remark

To argue by compactness you need to work where there is compactness: non smooth geometry (Ricci limits) enters into play.

Enlarging the study to $\text{RCD}(N-1, N)$ spaces it is possible to characterize rigidity:

Theorem (König, 15)

Let (X, d, μ) be an $\text{RCD}(N-1, N)$ m.m.s. with $\text{Vol} X^c / \mu(X, d, \mu) = N$. Then (X, d, μ) is isomorphic to a spherical suspension over an $\text{RCD}(N-2, N-1)$ m.m.s. (Y, d_Y, ν) .

Moreover, any eigenfunction associated to the first eigenvalue N is a multiple of $\cos(d_P)$, where P is a pole in the spherical suspension structure.

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Theorem (Colding-De Lellis)

Let (X, d, μ) be an $\text{RCD}(N-1, N)$ space with $\text{Per}(\mathcal{H}^1)(X, d, \mu) = N$. Then (X, d, μ) is isomorphic to a spherical suspension over an $\text{RCD}(N-2, N-1)$ metric (Y, d_Y, μ_Y) .

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Theorem (Colding-De Lellis)

Let (X, μ) be an $\text{RCD}(N-1, N)$ space with $\text{Per}(\mu) \neq \emptyset$. Then (X, μ) is isomorphic to a spherical suspension over an $\text{RCD}(N-2, N-1)$ metric space (Y, ν) .

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For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\text{RCD}(N-1, N)$ m.m.s. (X, d, m) and for any $u \in W^{1,2}(X, d, m)$ with $\|u\|_{L^2(m)} = 1$, $\int_X u dm = 0$ and

$$\int_X |\nabla u|^2 dm \leq N + \delta,$$

there exists $p \in X$ such that

$$\left\| u - \sqrt{N+1} \cos(d(p, \cdot)) \right\|_{L^2(m)} \leq \varepsilon.$$

The rigidity above is a RCD qualitative.

The question is whether RCD can be made quantitative (at least in some cases) and the approach is very much relying on entropy.

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The statement above is not quantitative.

The statement in [Peterson '99] can be made quantitative but requires strong PDE tools and the approach is very much relying on linearity.

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The framework

The **curvature-dimension condition** $CD(K, N)$ introduced by (Sturm '06) and (Lott-Villani '09) gives a way to talk about dimension upper bounds coupled with lower Ricci curvature bounds for metric measure spaces.

The **essentially non branching condition** was introduced in (Rajala-Sturm '14). A m.m.s. is e.n.b. if any optimal geodesic plan between probabilities a.c. with respect to the reference measure is concentrated on non branching geodesics.

We work in the framework of essentially non branching $CD(N - 1, N)$ metric measure spaces. This framework includes:

- **Weighted Riemannian manifolds**
- **Free limits**
- **CD $(N - 1, N)$ spaces**
- **Fractal models (with branching additional assumptions)**

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Main results

Theorem (Cavalletti-Mondino-S.'19)

There exists $C = C_N > 0$ such that, if (X, d, m) is an essentially non branching $CD(N-1, N)$ m.m.s., then

$$C_N(\pi - \text{diam}(X))^N \leq \lambda_{(X,d,m)}^{1,2} - N.$$

Theorem (Cavalletti-Mondino-S.'19)

For any $N \in \{1, \dots\}$ there exist $C(N) > 0$ and $\delta_N = \delta_N(N) > 0$ such that if (X, d, m) is an essentially non branching $CD(N-1, N)$ m.m.s. and $\mu \in \text{Lip}_1(X)$ is such that $\int \mu dm = 0$, $\int \mu^2 dm = 1$ and

$$\int \mu \sqrt{|\nabla \mu|^2} dm - N \leq \delta_N,$$

then there exists a probability measure $\nu \ll m$ on X such that

$$\left\| \mu - \sqrt{N + \int \mu^2 dm} \nu \right\|_{L^2(X, dm)} \leq C(N) \delta_N^{1/2}.$$

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$$\delta := \int_X |\nabla u|^2 dm - N \leq \delta_0,$$

then there exists a distinguished point $P \in X$ such that

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The tool: localization

Theorem (Cavalletti-Mondino '17)

Let (X, d, m) be an e.n.b. $CD(N-1, N)$ m.m.s. for some $N \in (1, \infty)$. Let $f: X \rightarrow \mathbb{R}$ be such that $\int_X f \, m = 0$. Then X can be written as the disjoint union of Z and T with T admitting a partition $\{X_q\}_{q \in Q}$ and a disintegration of $m \llcorner_T$, $\{\pi_q\}_{q \in Q}$ such that:

- For any m -measurable set $B \subset T$ it holds

$$m(B) = \int_Q \pi_q(B) \alpha(dq),$$

where α is a probability measure over Q defined on the quotient σ -algebra \mathcal{Q} .

- For α -almost every $q \in Q$, the set X_q is a geodesic and π_q is supported on it. Moreover $q \mapsto \pi_q$ is a $CD(N-1, N)$ disintegration.
- For α -a.s. $q \in Q$, $\int_{X_q} f \, \pi_q = 0$ and $f = 0$ m -a.e. in Z .

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- For α -almost every $q \in Q$, the set X_q is a geodesic and m_q is supported on it. Moreover $q \mapsto m_q$ is a $CD(N-1, N)$ disintegration.
- For α -a.e. $q \in Q$, $\int_{X_q} f m_q = 0$ and $f = 0$ m -a.e. in Z .

The tool: localization

Theorem (Cavalletti-Mondino '17)

Let (X, d, m) be an e.n.b. $CD(N-1, N)$ m.m.s. for some $N \in (1, \infty)$. Let $f : X \rightarrow \mathbb{R}$ be such that $\int_X f m = 0$. Then X can be written as the disjoint union of Z and \mathcal{T} with \mathcal{T} admitting a partition $\{X_q\}_{q \in Q}$ and a disintegration of $m \llcorner_{\mathcal{T}}$, $\{m_q\}_{q \in Q}$ such that:

- For any m -measurable set $B \subset \mathcal{T}$ it holds

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Remarks about the localization tool

Remark

Curvature-dimension bounds involve the coupling of distance and measure.

On weighted manifolds $(M^n, d_g, e^{-V} \text{vol}_g)$ the modified Ricci tensor

$$\text{Ric}_{N,n} = \text{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N - n}$$

is the relevant object to look at for the $\text{CD}(K, N)$ condition.

Localization simplifies as much as possible the metric space. The relevant information is encoded into the measure (Cavalletti-Mondino '17), (Klartag '17) after (Payne-Weinberger '60), (Gromov-Milman '87),... .

Weighted manifolds $(M^n, d_g, e^{-V} \text{vol}_g) \rightarrow \text{CD}(N-1, N)$ if and only if

$$\text{Ric}_{N,n} + \nabla^2 V \geq 0$$

Remarks about the localization tool

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Curvature-dimension bounds involve the coupling of distance and measure.

On **weighted manifolds** $(M^n, d_g, e^{-V} \text{vol}_g)$ the modified Ricci tensor

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Weighted manifolds $(M^n, d_g, e^{-V} \text{vol}_g) \rightarrow \text{CD}(N-1, N)$ via localization

$$\text{Ric}_{N,n} \geq K \Rightarrow \text{Ric}_{N-1, n} \geq K$$

Remarks about the localization tool

Remark

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Remark

Weighted intervals $(I, d_{\text{euc}}, h\mathcal{L}^1)$ are $\text{CD}(N-1, N)$ spaces if and only if

$$(h^{N-1})'' + h^{N-3} \leq 0.$$

Remarks about the localization tool

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Curvature-dimension bounds involve the coupling of distance and measure.

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Remark

Weighted intervals $(I, d_{\text{opt}}, h \mathcal{L}^1)$ are $\text{CD}(N-1, N)$ spaces if and only if

$$(h^2 - 1)^N + h^2 \leq 0.$$

Remarks about the localization tool

Remark

Curvature-dimension bounds involve the coupling of distance and measure.

On **weighted manifolds** $(M^n, d_g, e^{-V} \text{vol}_g)$ the modified Ricci tensor

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Remark

Weighted intervals $(I, d_{\text{eucl}}, h\mathcal{L}^1)$ are $\text{CD}(N - 1, N)$ spaces if and only if

$$\left(h^{\frac{1}{N-1}}\right)'' + h^{\frac{1}{N-1}} \leq 0.$$

Strategy of the proof: step 1

Apply the localization tool to the function u and get the corresponding disintegration.

Set $c_q^2 = \int_{X_q} u^2 dm_q$.

$$\begin{aligned}\delta(u) &= \int_X |\nabla u|^2 dm - N \geq \int_Q \left(\int_{X_q} \frac{|u'_q|^2}{c_q^2} dm_q \right) c_q^2 d\alpha - N \\ &= \int_Q \left[\int_{X_q} \left(\frac{|u'_q|^2}{c_q^2} - N \right) dm_q \right] c_q^2 d\alpha(dq) \\ &= \int_Q \delta(u_q) c_q^2 d\alpha(q).\end{aligned}$$

Disintegration of the Dirichlet form: the analysis of the 1d Dirichlet form on intervals

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Disintegration of the measure μ with respect to the projection $\pi: X \rightarrow Q$. [See also the proof of Theorem 1.10 in \[1\].](#)

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Remark:

Smallness of the deficit δ implies smallness of the 1d deficit $\delta(u_q)$ on average.

Strategy of the proof: step 1

Apply the localization tool to the function u and get the corresponding disintegration.

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Heuristic

Smallness of the deficit δ implies smallness of the 1d deficit $\delta(u_q)$ on average.

One dimensional results

The starting point arguing via localization is a study of the problem on $CD(N-1, N)$ weighted intervals.

Theorem

There exists a constant $C_N > 0$ such that, if (I, d_{eucl}, m) is a one dimensional $CD(N-1, N)$ m.m.s., then

$$C_N(\pi - \text{diam}(I))^N \leq \lambda_{(I, d_{\text{eucl}}, m)}^{1,2} - N.$$

Corollary (D'Ambrosio-Ottaviani-Segre)

For every $N \geq 1$ there exist constants $C = C(N) > 0$ and $\bar{D}_N = \bar{D}_N(N) > 0$ such that, if (I, d_{eucl}, m) is a $CD(N-1, N)$ m.m.s. and $\mu \in \mathcal{L}_1(I)$ satisfies $\int_I \mu dx = 0$, $\int_I \mu dx = 1$ and $\int_I |\mu|^2 dx - N \leq \bar{D}_N$, then

$$\min\left\{\int_I \mu^2 dx, \int_I |\mu| dx\right\} \leq C \sqrt{N} \sqrt{\int_I |\mu|^2 dx - N}.$$

One dimensional results

The starting point arguing via localization is a study of the problem on $CD(N-1, N)$ weighted intervals.

Theorem

There exists a constant $C_N > 0$ such that, if (I, d_{wcd}, m) is a one dimensional $CD(N-1, N)$ m.m.s., then

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Theorem (Cavalletti-Mondino-S. '19)

For every $N > 1$ there exist constants $C = C(N) > 0$ and $\delta_0 = \delta_0(N) > 0$ such that, if $([0, D], d_{eucl}, m)$ is a $CD(N - 1, N)$ m.m.s. and $u \in \text{Lip}(I)$ satisfies $\int u dm = 0$, $\int u^2 dm = 1$ and $\delta := \int |u'|^2 dm - N \leq \delta_0$, then

$$\min \left\{ \left\| u - \sqrt{N+1} \cos(\cdot) \right\|_{L^2(m)}, \left\| u + \sqrt{N+1} \cos(\cdot) \right\|_{L^2(m)} \right\} \leq C \delta^{\min\{1/2, 1/N\}}.$$

One dimensional results

The starting point arguing via localization is a study of the problem on $CD(N-1, N)$ weighted intervals.

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Strategy of the proof: step 2

We fix $0 < \beta < 1$ and let Q_ℓ be the set of “long rays” q for which $\delta(u_q) \leq \delta^\beta$.
By Chebyshev there are many long rays (w.r.t. $c_q^2 \alpha(dq)$).

Theorem (1)

There exist $C(N) > 0$ and $\alpha(N) > 0$ such that

$$\int_{Q_\ell} \left| c_q - \frac{1}{\alpha(Q_\ell)} \int_{Q_\ell} c_q d\alpha(q) \right|^2 d\alpha(q) \leq C(N) \delta^{\alpha(N)}.$$

Theorem (2)

There exist $C(N) > 0$ and $\alpha(N) > 0$ such that

$$\alpha(Q_\ell) \geq 1 - C(N) \delta^{\alpha(N)}.$$

Strategy of the proof: step 2

We fix $0 < \beta < 1$ and let Q_ℓ be the set of “long rays” q for which $\delta(u_q) \leq \delta^\beta$.
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There exist $C(N) > 0$ and $\alpha(N) > 0$ such that

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We fix $0 < \beta < 1$ and let Q_ℓ be the set of “long rays” q for which $\delta(u_q) \leq \delta^\beta$. By Chebyshev there are many long rays (w.r.t. $c_q^2 \alpha(dq)$).

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Comments about Step 2

Step 2 is the most technical part of the proof. It requires new ideas with respect to (Cavalletti-Maggi-Mondino '17).

Isoperimetric inequality: perimeter is minimized by spheres (local).

Spectral gap inequality: energy is minimized with mean and L^2 -norm fixed. Mean is fixed by localization, it is necessary to control the L^2 -norms of the restrictions to rays.

The reason of the cylindrical shape comes from the control of the energy associated to the low order L^2 -norm.

Comments about Step 2

Step 2 is the most technical part of the proof. It requires new ideas with respect to (Cavalletti-Maggi-Mondino '17).

Remark

Isoperimetric inequality: perimeter is minimized at volume fixed.

Spectral gap inequality: energy is minimized with mean and L^2 -norm fixed. Mean is fixed by localization, it is necessary to control the L^2 -norms of the restrictions to rays.

The mission of step 2 is to control the energy of the gradient restricted to rays (how small L^2 -norm).

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Theorem 1.10 of Cavalletti-Maggi-Mondino '17

Proposition 1.11 of Cavalletti-Maggi-Mondino '17

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Heuristic

The variation of $g \mapsto c_g$ is small since the components of the gradient transversal to rays have small L^2 -norm.

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Hour 6/6

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Heuristic

The variation of $q \mapsto c_q$ is small since the components of the gradient transversal to rays have small L^2 -norm.

About the proof

The heuristic above presents two main difficulties:

- seems to rely on infinitesimal hilbertianity;
- seems to rely on good properties of the quotient Ω .

The proofs of **Theorem 1** and **Theorem 2** are obtained applying the Poincaré inequality on small balls centred at suitably chosen *poles*.

we estimate the variance of u on the ball in terms of the variance of $q \rightarrow q_0$ and of the measure of long rays;

we estimate the energy of u on the ball;

we get the sought estimate by Poincaré inequality.

About the proof

The heuristic above presents two main difficulties:

- seems to rely on infinitesimal hilbertianity;
- seems to rely on *good* properties of the quotient Q .

The proofs of **Theorem 1** and **Theorem 2** are obtained applying the Poincaré inequality on small balls centred at suitably chosen *poles*.

we estimate the volume of u on the ball in terms of the volume of $u|_{\partial B}$ and of the measure of long rays;

we estimate the energy of u on the ball;

we get an upper estimate by Poincaré inequality.

About the proof

The heuristic above presents two main difficulties:

- seems to rely on infinitesimal hilbertianity;
- seems to rely on *good* properties of the quotient Q .

The proofs of **Theorem 1** and **Theorem 2** are obtained applying the Poincaré inequality on small balls centred at suitably chosen *poles*.

- we estimate the volume of u on the ball in terms of the volume of u^2 and of the measure of $\log |u|$;
- we estimate the energy of u on the ball;
- we estimate $\int u^2$ by Poincaré inequality.

About the proof

The heuristic above presents two main difficulties:

- seems to rely on infinitesimal hilbertianity;
- seems to rely on *good* properties of the quotient Q .

The proofs of **Theorem 1** and **Theorem 2** are obtained applying the **Poincaré inequality** on small balls centred at suitably chosen *poles*.

About the proof

The heuristic above presents two main difficulties:

- seems to rely on infinitesimal hilbertianity;
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The proofs of **Theorem 1** and **Theorem 2** are obtained applying the **Poincaré inequality** on small balls centred at suitably chosen *poles*.

- we estimate the variance of u on the ball in terms of the variance of $q \mapsto c_q$ and of the measure of long rays;
- we estimate the energy of u on the ball;
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Strategy of the proof: step 3

- a) By Step 1 u is "concentrated" on long rays. In particular, there is a ray with small deficit: we get the diameter estimate.
- b) by Theorem 1, u is "almost constant" on Ω .
- c) combining (a) and (b) with Theorem 2, u is almost constant and equal to $\frac{1}{2} \text{diam}(\Omega)$.
- d) combining (c) with

the isoperimetric inequality, we get the result.

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- c) combining (a) and (b) with **Theorem 2.2**, u almost constant and small deficit on a ray;
- d) combine (c) with **Theorem 2.1**.

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- d) combine (c) with **Theorem 3**.

→ **conclude the proof** (see next slide)

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→ **conclude the proof** (by contradiction)

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Remarks and open problems

Main improvements with respect to the literature on the topic:

- explicit behaviour in the quantitative inequality;
- extension to the non-analytic framework;
- approach relying on the variational characterization rather than on the spectral theory.

Open problems:

- sharpness of the exponent;
- improving the non-linear case first eigenvalue of the p -Laplacian;
- increasing the class of elliptic boundary conditions.

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Main improvements with respect to the literature on the topic:

- explicit behaviour in the quantitative inequality;
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Open problems:

- sharpness of the constant;
- improving the constant for general convex sets of the plane;
- extending the case of \mathbb{R}^n to general convex sets.

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Thank you for the attention!