

# Structure theory of $\text{RCD}(K, N)$ spaces via $\delta$ -splitting maps<sup>1</sup>

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<sup>1</sup>Based on "Rectifiability of  $\text{RCD}(K, N)$  spaces via  $\delta$ -splitting maps", accepted by Ann. Acad. Sci. Fenn. Math. (2020), joint with E. Brué and E. Pasqualetto.

# Introduction

RCD spaces are (possibly) non smooth **metric measure spaces** with **Ricci curvature** bounded from below and **dimension** bounded from above, in synthetic sense.

## Structure theory

Addressing the regularity for these spaces and estimating the size of singularities.

In [Brué-Pasqualetto-S. '20] we give simplified proofs of the known results about the structure of  $\text{RCD}(K, N)$  spaces relying on  $\delta$ -splitting maps.

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# Outline

- 1 Ricci curvature lower bounds
- 2  $\text{RCD}(K, N)$  spaces
- 3 Structure theory of  $\text{RCD}(K, N)$  spaces
- 4  $\delta$ -splitting maps
- 5 Structure theory via  $\delta$ -splitting maps
- 6 Further developments

# Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map  $\psi : M \rightarrow \mathbb{R}$  and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if  $\dot{\gamma} := \frac{d}{dt} T_t(x)$  and  $\mathcal{J}(t) = \det DT_t(x)$  is the **volume element**,

$$\frac{d^2}{dt^2} (\mathcal{J}(t))^{1/n} + \frac{\text{Ric}(\dot{\gamma}, \dot{\gamma})}{n} \mathcal{J}^{1/n} \leq 0 \quad (1)$$

and

$$\Delta \frac{|\nabla\psi|^2}{2} - \nabla\psi \cdot \nabla\Delta\psi \geq \frac{(\Delta\psi)^2}{n} + \text{Ric}(\nabla\psi, \nabla\psi). \quad (2)$$

These two inequalities are equivalent to the  
Lichnerowicz inequality on Ricci curvature.

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# Ricci curvature bounds

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Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

→ Bishop-Gromov inequality on monotonicity of volume ratios

→ Cheeger-Croke  $\pi_1$ -Betti theorem

→ Li-Yau heat kernel bounds

→ spectral gap and diameter estimates

→ Li-Yang-Yang isoperimetric inequality

## Dimensional bounds

The Cheeger-Croke  $\pi_1$ -Betti theorem (Ricci curvature bounded below, upper bound on dimension) can be extended to complete bounded below  $K$ -Ricci manifolds with the lower-dimensional analogue

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- Cheeger-Colding-Cheung theory;

- Heat kernel bounds;

- Spectral theory of Laplacian eigenvalues;

- Geometric measure theory.

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## Theorem ([Gromov])

*The class  $\mathcal{M}_{N,D,K}$  of smooth Riemannian manifolds with dimension  $N$ , diameter bounded above by  $D$  and Ricci curvature bounded below by  $K$  is precompact w.r.t. the Gromov-Hausdorff topology.*

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# The quest for synthetic notions, I

## Question

How do Riemannian manifolds in  $\mathcal{M}_{N,D,K}$  look like?

The question motivated the theory of **Ricci limits**, limits in the (pm)GH topology of manifolds in  $\mathcal{M}_{N,D,K}$ , initiated by **Cheeger-Colding** in the Nineties.

• Riemannian manifolds with bounded Ricci curvature and volume are bounded from below by uniform  $C^2$ -estimates for the Riemann curvature tensor (Cheeger-Colding, 1997).

• Riemannian manifolds with lower Ricci curvature bounds do not enjoy uniform  $C^2$ -estimates by harmonic functions (Cheeger-Colding, 1997).

The study of Ricci limits improves our knowledge of  $\mathcal{M}_{N,D,K}$ .

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For Riemannian manifolds with bounded Ricci curvature and volume, Cheeger-Colding proved very powerful estimates for the Ricci curvature (see [CC97]).

For Riemannian manifolds with lower Ricci curvature bounds, Cheeger-Colding proved  $C^2$ -estimates for harmonic functions (see [CC97]).

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# The quest for synthetic notions, II

The **synthetic** treatment of lower Ricci curvature bounds stems from the following

## Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

Curvature  $\geq K$  theory also deal with metric measure spaces. The limit theory is RCD(K, N).

Curvature  $\geq K$  theory is necessary to recover the Bishop-Gromov volume comparison in the limit.

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Starting with the theory of lower Ricci curvature bounds on Alexandrov spaces, the theory of Ricci limit spaces was developed by Colding and Perelman.

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## Calculus tools

On a m.m.s.  $(X, d, \mathfrak{m})$  we introduce the *Cheeger energy*  $\text{Ch} : L^2(X, \mathfrak{m}) \rightarrow [0, +\infty]$  by

$$\text{Ch}(f) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_X (\text{lip} f_n)^2 d\mathfrak{m} : f_n \rightarrow f \text{ in } L^2(X, \mathfrak{m}), f_n \in \text{Lip}(X, d) \right\}.$$

- There exists a *minimal relaxed gradient*  $|\nabla f|$  such that  $\text{Ch}(f) = \int_X |\nabla f|^2 d\mathfrak{m}$  for any  $f \in \{\text{Ch} < +\infty\}$ ;
- it is possible to define a *heat flow*  $P_t$  and a *laplacian*  $\Delta$  as the gradient flow of  $\frac{1}{2} \text{Ch}$  over  $L^2(X, \mathfrak{m})$  and its infinitesimal generator, respectively.

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- There exists a *minimal relaxed gradient*  $|\nabla f|$  such that  $\text{Ch}(f) = \int_X |\nabla f|^2 dm$  for any  $f \in \{\text{Ch} < +\infty\}$ ;
- it is possible to define a *heat flow*  $P_t$  and a *laplacian*  $\Delta$  as the gradient flow of  $\frac{1}{2} \text{Ch}$  over  $L^2(X, m)$  and its infinitesimal generator, respectively.

### Definition

A m.m.s.  $(X, d, m)$  is *infinitesimally Hilbertian* if  $\text{Ch}$  is a quadratic form on  $L^2(X, m)$ .

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On any infinitesimally Hilbertian m.m.s.  $P_t$  and  $\Delta$  are linear.

## Calculus tools

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# RCD( $K, N$ ) spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Gardero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Bavand Savard-Lou-Villani '07] for the proposal of the  $\text{RCD}(K, N)$  Curvature-Dimension condition on metric measure spaces;
- [Ambrosio-Ceccati-Villani '08], [Ambrosio-Ceccati-Villani '09] for the natural extension assumption:

**Definition 1.1** ( $\text{RCD}(K, N)$  spaces)

For any  $K \in \mathbb{R}$  and  $1 \leq N < \infty$  we say that  $(X, \mu) \in \text{RCD}(K, N)$  if it is an infinitesimally Hilbertian  $\text{CD}(K, N)$  space.

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- [Sturm '06] and [Lott-Villani '07], for the proposal of the CD( $K, N$ ) Curvature-Dimension condition on metric measure spaces;
- [Ambrosio-Colesanti-Mantegazza '07], [Ambrosio '08] for the proposal of the RCD( $K, N$ ) condition.

Definition 1.1 (RCD( $K, N$ ))

For any  $K \in \mathbb{R}$  and  $N \in \mathbb{N} \cup \{\infty\}$  we say that  $(X, d, \mu) \in \mathbf{RCD}(K, N)$  if it is an  $\mathbf{CD}(K, N)$  space and  $\mathbf{CD}(K, N)$  is tight.

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# The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem (RCD<sup>\*</sup>(K, N) spaces)

A m.m.s.  $(X, d, m)$  is RCD<sup>\*</sup>(K, N) if:

- $m(B_r(x)) \leq \alpha_1 \exp(\alpha_2 r^2)$  for some  $x \in X$  and constants  $\alpha_1, \alpha_2 > 0$ ;
- $d$  is infinitesimally Hilbertian;
- it satisfies the Sobolev to Lipschitz property;
- $\Delta_{\frac{1}{2}} |\nabla f|^2 - \nabla f \cdot \nabla \Delta f \geq \frac{K|f|}{\sqrt{N}} + K|\nabla f|^2$ , for any  $f$  in a class of test functions.

RCD<sup>\*</sup>(K, N) is equivalent to RCD(K, N) if  $m(X) < \infty$  (Theorem 10)

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# Remarks & motivations

- The splitting theorem ([Cheeger-Colding], [Cheeger-Colding]) holds for  $\text{RCD}(0, N)$  spaces [Gigli '13]. The split factor is  $\text{RCD}(0, N - 1)$ ;
- a cone is  $\text{RCD}(0, N)$  if and only if the cross section is  $\text{RCD}(N - 2, N - 1)$  [Ketterer '13];
- quotients of  $\text{RCD}^*(K, N)$  spaces under isometric group actions are  $\text{RCD}^*(K, N)$  spaces [Colding-Cheeger 2011, Gigli '13];
- profinite limits of manifolds in  $\mathcal{M}_{\text{RCD}}(K, N)$  are  $\text{RCD}(K, N)$  spaces;
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# Tangent spaces

## Question

How regular is an  $\text{RCD}(K, N)$  space?

## Definition (Tangent cone)

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  and  $x \in X$  we let  $\text{Tan}_x(X, d, m)$  be the set of all pmGH limits

$$(Y, d_Y, m_Y, y) = \lim_{r \rightarrow \infty} (X, r_i^{-1} d, m_{r_i}^x, x),$$

where  $r_i \downarrow 0$  and  $m_{r_i}^x = c_{r_i}^x m$  for some normalizing constant  $c_{r_i}^x > 0$ .

$$\text{e.g. } \text{Tan}_x(\mathbb{R}^n, d_{\text{eucl}}) = (\mathbb{R}^n, d_{\text{eucl}}, m_x), \text{ and } \text{Tan}_x(\mathbb{R}^n, d_{\text{eucl}}) = \mathbb{R}^n \text{ for } x \in \mathbb{R}^n.$$

• The tangent cone to a metric cone at its tip is the cone itself.

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- If  $(X, d, m) = (M^n, d_g, \text{vol}_g)$ , then  $\text{Tan}_x(X, d, m) = \{(\mathbb{R}^n, d_{\text{eucl}}, c_n \mathcal{L}^n, 0^n)\}$  for any  $x \in X$ ;
- the tangent cone to a metric cone at its tip is the cone itself.

# Tangent spaces

## Question

How regular is an  $\text{RCD}(K, N)$  space?

## Definition (Tangent cone)

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  and  $x \in X$  we let  $\text{Tan}_x(X, d, m)$  be the set of all pmGH limits

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# Structure theory: the state of the art

## Definition ( $k$ -regular set)

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. For any  $1 \leq k \leq N$  let

$$\mathcal{R}_k := \{x \in X : \text{Tan}_x(X, d, m) = \{(\mathbb{R}^k, d_{\text{eucd}}, \alpha_k \mathcal{L}^k, 0^k)\}\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

## Theorem

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s. and let  $\mathcal{R}_k$  be the set of  $k$ -regular points. Then

$$m\left(X \setminus \bigcup_{k=1}^N \mathcal{R}_k\right) = 0.$$

Furthermore, for any  $1 \leq k \leq N$  the  $k$ -regular set  $\mathcal{R}_k$  is  $(m, k)$ -rectifiable in the sense that  $\mathcal{R}_k = \mathcal{R}_k^{\text{reg}}$  for some density  $\theta \in L^1_{\text{loc}}(\mathbb{R}^k \llcorner \mathcal{R}_k)$ .

$(m, k)$ -rectifiable means that, up to an  $m$ -negligible set, we can cover it with a countable union of biLipschitz images of subsets of  $\mathbb{R}^k$ .

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# Comments

## Remark

In [Brué-S. 18] we proved that  $\text{RCD}(K, N)$  spaces have constant dimension in the almost everywhere sense. This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits: the introduction of the notion of  $\text{RCD}^*(K, N)$  spaces at the time of [Colding-Naber '12] and the use of a key lemma of [Colding-Naber '12] on almost everywhere constant dimension.

## Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

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Different approach needed with respect to the structure theory for Ricci limits:

• Different notion of measure on  $\text{RCD}(K, N)$  spaces as the limit of measures on manifolds

• Different notion of a key lemma of [Colding-Naber '12] on almost everywhere

## Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

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Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];

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Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

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# Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on  $\text{RCD}(K, \infty)$  spaces has been developed. See also [Savaré '14] and [Sturm '15].

## Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.

Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\text{Hess}f|^2 dm \leq C_{K,N} \left( \int_{B_2(x)} (\Delta f)^2 dm + \inf_{m \in \mathbb{R}} \int_{B_2(x)} \|\nabla f\|^2 - m dm \right) - K \int_{B_2(x)} |\nabla f|^2.$$

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# Harmonic $\delta$ -splitting maps

## Definition ( $\delta$ -splitting map)

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s. and  $B_r(x) \subset X$ . We say that  $u: B_r(x) \rightarrow \mathbb{R}^k$  is a  $\delta$ -splitting map provided:

- $u$  has harmonic and  $C_W$ -Lipschitz components,
- $\frac{r^2}{\text{vol}(B_r(x))} \int_{B_r(x)} |\text{Hess}u|^2 dm \leq \delta,$
- $\frac{r^2}{\text{vol}(B_r(x))} \int_{B_r(x)} |\nabla u_k \cdot \nabla u_k - \delta_{kk}| dm \leq \delta.$

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Cheeger-Colding '97] we give new proofs of the structure theory for  $\text{RCD}(K, N)$  spaces using  $\delta$ -splitting maps to replace, as in the Cheeger-Colding theory,

This tool allows to extend the theory to positive codimension.

# Harmonic $\delta$ -splitting maps

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(i)  $u$  has harmonic and  $C_N$ -Lipschitz components;

$$\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\text{Hess} u|^2 d\mathfrak{m} \leq \delta;$$

$$\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| d\mathfrak{m} \leq \delta.$$

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$$b) \frac{r^2}{\omega(B_r(x))} \int_{B_r(x)} |\text{Hess}u|^2 dm \leq \delta;$$

$$c) \frac{r^2}{\omega(B_r(x))} \int_{B_r(x)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| dm \leq \delta.$$

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- c)  $\frac{1}{m(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| dm \leq \delta$ .

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

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# The intuition

Suppose  $X = \mathbb{R}^k \times Y$  and let  $u := (x_1, \dots, x_k)$  be the canonical coordinates on the factor  $\mathbb{R}^k$ .

They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

## Remark

Harmonic  $\delta$ -splitting maps are harmonic approximations in the  $W^{1,2}$ -sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

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# $\delta$ -splitting implies $\varepsilon$ -isometry

If at a certain location a map is  $\delta$ -splitting at all sufficiently small scales, then any tangent is close to a split space.

## Theorem

Let  $N > 1$  be given. Then for any  $\varepsilon > 0$  there exists  $\delta = \delta_{N,\varepsilon} > 0$  such that the following holds.

If  $(X, d, m)$  is an  $\text{RCD}(K, N)$  space,  $x \in X$ , and there exists a map  $u: B_r(x) \rightarrow \mathbb{R}^k$  such that  $u: B_s(x) \rightarrow \mathbb{R}^k$  is a  $\delta$ -splitting map for all  $s < r$ , then for any  $(Y, \varrho, n, \gamma) \in \text{Tan}_x(X, d, m)$

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If a space is  $\varepsilon$ -close to a split space with Euclidean factor  $\mathbb{R}^k$ , then there is a  $(k, \delta)$ -splitting map.

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# Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13] )

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s. Then for  $m$ -a.e.  $x \in X$  there exists  $1 \leq k \leq N$  such that

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Proven by recursive application of the splitting theorem, relying on the principle that tangents of tangents are tangents (see [1], [2], [3]).

We also obtain an independent proof of the existence of regular tangents via a different strategy.

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The starting point is the following

**Theorem** ([Gigli-Mondino-Rajala '13] )

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. Then for  $m$ -a.e.  $x \in X$  there exists  $1 \leq k \leq N$  such that

$$(\mathbb{R}^k, d_{\text{eucl}}, c_k \mathcal{L}^k, 0^k) \in \text{Tan}_x(X, d, m).$$

## Remark

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the [existence of regular tangents](#) via a different strategy.

# Propagation of regularity



# Why using harmonic $\delta$ -splitting maps

## Remark

In the study of singular sets on non collapsed Ricci limits, harmonic  $\delta$ -splitting maps verify much better estimates than general  $\delta$ -splitting maps, see [Cheeger-Jiang-Naber '18].

Also on general  $\text{RCD}(K, N)$  spaces harmonic  $\delta$ -splitting maps turn to be useful.

## Remark

In [Drué-Pasqualetto-S. '19] they are used to prove rectifiability of reduced boundaries for sets of finite perimeter.

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# Possible directions

Infinite dimensional case  
Non linear case

Thank you for your attention!