Structure theory of $\mathrm{RCD}(K, N)$ spaces via δ-splitting maps[1]

Daniele Semola

Scuola Normale Superiore, Pisa
daniele.semola@sns.it

[1] Based on "Rectifiability of $\mathrm{RCD}(K, N)$ spaces via δ-splitting maps", accepted by Ann. Acad. Sci. Fenn. Math. (2020), joint with E. Brué and E. Pasqualetto.

Introduction

RCD spaces are (possibly) non smooth metric measure spaces with Ricci curvature bounded from below and dimension bounded from above, in synthetic sense.

Structure theory

Addressing the regularity for these spaces and estimating the size of singularities.

In [Brué-Pasqualetto-S. '20] we give simplified proofs of the known results about the structure of $\mathrm{RCD}(K, N)$ spaces relying on δ-splitting maps.

Introduction

RCD spaces are (possibly) non smooth metric measure spaces with Ricci curvature bounded from below and dimension bounded from above, in synthetic sense.

Structure theory

Addressing the regularity for these spaces and estimating the size of singularities.

In [Brué-Pasqualetto-S. '20] we give simplified proofs of the known results about the structure of $\mathrm{RCD}(K, N)$ spaces relying on δ-splitting maps.

Introduction

RCD spaces are (possibly) non smooth metric measure spaces with Ricci curvature bounded from below and dimension bounded from above, in synthetic sense.

Structure theory

Addressing the regularity for these spaces and estimating the size of singularities.

In [Brué-Pasqualetto-S. '20] we give simplified proofs of the known results about the structure of $\mathrm{RCD}(K,N)$ spaces relying on δ-splitting maps.

Introduction

RCD spaces are (possibly) non smooth metric measure spaces with Ricci curvature bounded from below and dimension bounded from above, in synthetic sense.

Structure theory

Addressing the regularity for these spaces and estimating the size of singularities.

In [Brué-Pasqualetto-S. '20] we give simplified proofs of the known results about the structure of $\mathrm{RCD}(K, N)$ spaces relying on δ-splitting maps.

Outline

Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \qquad (1)$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \qquad (2)$$

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is an Eulerian perspective on Ricci curvature.

Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \tag{1}$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{2}$$

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \tag{1}$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{2}$$

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \tag{1}$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{2}$$

Remark

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \tag{1}$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{2}$$

Remark

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \tag{1}$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{2}$$

Remark

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios,
- Cheeger-Gromoll splitting theorem,
- Li-Yau heat kernel bounds,
- spectral gap and diameter estimates,
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- spectral gap and diameter estimates;
- Lévy-Gromov isoperimetric inequality.

Theorem ([Gromov])

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2 bounds for the Riemann curvature tensor ([Jiang-Naber '16]).
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1 estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, I

Question

How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2-bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- Riemannian manifolds with lower Ricci curvature bounds do not verify uniform C^1-estimates for harmonic functions ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Remark

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Remark

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Remark

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Remark

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

The quest for synthetic notions, II

The synthetic treatment of lower Ricci curvature bounds stems from the following

Question

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Remark

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathsf{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathsf{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathsf{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n \to \infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathsf{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathsf{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathsf{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathsf{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathsf{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathsf{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathsf{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathsf{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathsf{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathsf{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathsf{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathsf{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

Remark

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathrm{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathrm{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathrm{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathrm{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

Remark

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

Calculus tools

On a m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ we introduce the *Cheeger energy* $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \int_X (\mathrm{lip} f_n)^2 \mathrm{d}\mathfrak{m} : \quad f_n \to f \text{ in } L^2(X, \mathfrak{m}), \quad f_n \in \mathrm{Lip}(X, \mathrm{d}) \right\}.$$

- There exists a *minimal relaxed gradient* $|\nabla f|$ such that $\mathsf{Ch}(f) = \int_X |\nabla f|^2 \mathrm{d}\mathfrak{m}$ for any $f \in \{\mathsf{Ch} < +\infty\}$;
- it is possible to define a *heat flow* P_t and a *laplacian* Δ as the gradient flow of $\frac{1}{2}\mathsf{Ch}$ over $L^2(X, \mathfrak{m})$ and its infinitesimal generator, respectively.

Definition

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if Ch is a quadratic form on $L^2(X, \mathfrak{m})$.

Remark

On any infinitesimally Hilbertian m.m.s. P_t and Δ are linear.

RCD(K, N) spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the CD(K, N) Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption.

Definition (RCD(K, N) m.m.s.)

For any $K \in \mathbb{R}$ and $1 \leq N < \infty$ we say that (X, d, m) is RCD(K, N) if it is an infinitesimally Hilbertian CD(K, N) m.m.s..

$\mathrm{RCD}(K, N)$ spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption.

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \leq N < \infty$ we say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s..

$\mathrm{RCD}(K, N)$ spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption:

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \leq N < \infty$ we say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s..

$\mathrm{RCD}(K, N)$ spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption:

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \le N < \infty$ we say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s..

$\mathrm{RCD}(K, N)$ spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption:

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \leq N < \infty$ we say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s..

$\mathrm{RCD}(K, N)$ spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption:

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \le N < \infty$ we say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s..

$\mathrm{RCD}(K, N)$ spaces

After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption:

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \leq N < \infty$ we say that $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s..

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some $x \in X$ and constants $c_1, c_2 > 0$;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2} |\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K |\nabla f|^2$, *for any f in a class of test functions.*

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X,\mathrm{d},\mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some* $x \in X$ *and constants* $c_1, c_2 > 0$;
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2}|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K|\nabla f|^2$*, for any* f *in a class of test functions.*

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some $x \in X$ and constants $c_1, c_2 > 0$;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2}|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K|\nabla f|^2$*, for any f in a class of test functions.*

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$ thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some* $x \in X$ *and constants* $c_1, c_2 > 0$*;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2} |\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K |\nabla f|^2$*, for any* f *in a class of test functions.*

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X,\mathrm{d},\mathfrak{m})$ *is* $\mathrm{RCD}^*(K,N)$ *if:*

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some* $x \in X$ *and constants* $c_1, c_2 > 0$*;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2}|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K|\nabla f|^2$*, for any* f *in a class of test functions.*

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

- $\mathfrak{m}(B_r(x)) \leq c_1 \exp(c_2 r^2)$ *for some* $x \in X$ *and constants* $c_1, c_2 > 0$;
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2}|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \geq \frac{(\Delta f)^2}{N} + K|\nabla f|^2$, *for any f in a class of test functions.*

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K, N)$ spaces)

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K, N)$ if:

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some $x \in X$ and constants $c_1, c_2 > 0$;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2} |\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K |\nabla f|^2$, *for any f in a class of test functions.*

$\mathrm{RCD}^*(K, N)$ is equivalent to $\mathrm{RCD}(K, N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K,N)$ spaces)

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

- $\mathfrak{m}(B_r(x)) \leq c_1 \exp(c_2 r^2)$ *for some* $x \in X$ *and constants* $c_1, c_2 > 0$;
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2}|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \geq \frac{(\Delta f)^2}{N} + K|\nabla f|^2$, *for any* f *in a class of test functions.*

Remark

$\mathrm{RCD}^*(K,N)$ is equivalent to $\mathrm{RCD}(K,N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K, N)$ spaces)

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K, N)$ if:

- $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ *for some $x \in X$ and constants $c_1, c_2 > 0$;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2}|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K|\nabla f|^2$, *for any f in a class of test functions.*

Remark

$\mathrm{RCD}^*(K, N)$ is equivalent to $\mathrm{RCD}(K, N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

The Eulerian approach

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

Theorem ($\mathrm{RCD}^*(K, N)$ spaces)

A m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K, N)$ if:

- $\mathfrak{m}(B_r(x)) \leq c_1 \exp(c_2 r^2)$ *for some* $x \in X$ *and constants* $c_1, c_2 > 0$*;*
- *it is infinitesimally Hilbertian;*
- *it satisfies the Sobolev to Lipschitz property;*
- $\Delta \frac{1}{2} |\nabla f|^2 - \nabla f \cdot \nabla \Delta f \geq \frac{(\Delta f)^2}{N} + K |\nabla f|^2$, *for any* f *in a class of test functions.*

Remark

$\mathrm{RCD}^*(K, N)$ is equivalent to $\mathrm{RCD}(K, N)$ if $\mathfrak{m}(X) < \infty$, thanks to [Cavalletti-Milman '16].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-García-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{K,D,N}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{N,D,K}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{N,D,K}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{N,D,K}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{N,D,K}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{N,D,K}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Remarks & motivations

- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
- a cone is $\mathrm{RCD}(0, N)$ if and only if the cross section is $\mathrm{RCD}(N-2, N-1)$ [Ketterer '13];
- quotients of $\mathrm{RCD}^*(K, N)$ spaces under suitable group actions are $\mathrm{RCD}^*(K, N)$ spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
- pmGH limits of manifolds in $\mathcal{M}_{N,D,K}$ are $\mathrm{RCD}(K, N)$ spaces;
- Alexandrov spaces with dimension n and curvature bounded below by k equipped with the Hausdorff measure $\mathcal{H}^n$ are $\mathrm{RCD}(k(n-1), n)$ spaces [Petrunin '11], [Zhang-Zhu '10];
- the cone over $\mathbb{RP}^2$ is not limit of smooth Riemannian manifolds with nonnegative Ricci curvature and dimension less than 3 [Simon '12], [Simon-Topping '17].

Tangent spaces

Question

How regular is an $\mathrm{RCD}(K,N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathrm{d},\mathfrak{m})$ and $x\in X$ we let $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})$ be the set of all pmGH limits

$$(Y,\mathrm{d}_Y,\mathfrak{m}_Y,y)=\lim_{i\to\infty}(X,r_i^{-1}\mathrm{d},\mathfrak{m}_{r_i}^x,x),$$

where $r_i\downarrow 0$ and $\mathfrak{m}_{r_i}^x=c_{r_i}^x\mathfrak{m}$ for some normalizing constant $c_{r_i}^x>0$.

- If $(X,\mathrm{d},\mathfrak{m})=(M^n,\mathrm{d}_g,\mathrm{vol}_g)$, then $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})=\{(\mathbb{R}^n,\mathrm{d}_{eucl},c_n\mathscr{L}^n,0^n)\}$ for any $x\in X$;
- the tangent cone to a metric cone at its tip is the cone itself.

Tangent spaces

Question

How regular is an $\mathrm{RCD}(K,N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathrm{d},\mathfrak{m})$ and $x \in X$ we let $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})$ be the set of all pmGH limits

$$(Y,\mathrm{d}_Y,\mathfrak{m}_Y,y) = \lim_{i\to\infty}(X,r_i^{-1}\mathrm{d},\mathfrak{m}^x_{r_i},x),$$

where $r_i \downarrow 0$ and $\mathfrak{m}^x_{r_i} = c^x_{r_i}\mathfrak{m}$ for some normalizing constant $c^x_{r_i} > 0$.

- If $(X,\mathrm{d},\mathfrak{m}) = (M^n,\mathrm{d}_g,vol_g)$, then $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m}) = \{(\mathbb{R}^n,\mathrm{d}_{eucl},c_n\mathscr{L}^n,0^n)\}$ for any $x \in X$;
- the tangent cone to a metric cone at its tip is the cone itself.

Tangent spaces

Question

How regular is an $\mathrm{RCD}(K,N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathrm{d},\mathfrak{m})$ and $x\in X$ we let $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})$ be the set of all pmGH limits

$$(Y,\mathrm{d}_Y,\mathfrak{m}_Y,y)=\lim_{i\to\infty}(X,r_i^{-1}\mathrm{d},\mathfrak{m}^x_{r_i},x),$$

where $r_i\downarrow 0$ and $\mathfrak{m}^x_{r_i}=c^x_{r_i}\mathfrak{m}$ for some normalizing constant $c^x_{r_i}>0$.

- If $(X,\mathrm{d},\mathfrak{m})=(M^n,\mathrm{d}_g,vol_g)$, then $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})=\{(\mathbb{R}^n,\mathrm{d}_{eucl},c_n\mathscr{L}^n,0^n)\}$ for any $x\in X$;
- the tangent cone to a metric cone at its tip is the cone itself.

Tangent spaces

Question

How regular is an $\mathrm{RCD}(K,N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathrm{d},\mathfrak{m})$ and $x\in X$ we let $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})$ be the set of all pmGH limits

$$(Y,\mathrm{d}_Y,\mathfrak{m}_Y,y)=\lim_{i\to\infty}(X,r_i^{-1}\mathrm{d},\mathfrak{m}^x_{r_i},x),$$

where $r_i\downarrow 0$ and $\mathfrak{m}^x_{r_i}=c^x_{r_i}\mathfrak{m}$ for some normalizing constant $c^x_{r_i}>0$.

- If $(X,\mathrm{d},\mathfrak{m})=(M^n,\mathrm{d}_g,vol_g)$, then $\mathrm{Tan}_x(X,\mathrm{d},\mathfrak{m})=\{(\mathbb{R}^n,\mathrm{d}_{eucl},c_n\mathscr{L}^n,0^n)\}$ for any $x\in X$;
- the tangent cone to a metric cone at its tip is the cone itself.

Tangent spaces

Question

How regular is an $\mathrm{RCD}(K, N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathrm{d}, \mathfrak{m})$ and $x \in X$ we let $\mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m})$ be the set of all pmGH limits

$$(Y, \mathrm{d}_Y, \mathfrak{m}_Y, y) = \lim_{i \to \infty} (X, r_i^{-1}\mathrm{d}, \mathfrak{m}^x_{r_i}, x),$$

where $r_i \downarrow 0$ and $\mathfrak{m}^x_{r_i} = c^x_{r_i}\mathfrak{m}$ for some normalizing constant $c^x_{r_i} > 0$.

- If $(X, \mathrm{d}, \mathfrak{m}) = (M^n, \mathrm{d}_g, vol_g)$, then $\mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}) = \{(\mathbb{R}^n, \mathrm{d}_{eucl}, c_n\mathscr{L}^n, 0^n)\}$ for any $x \in X$;
- the tangent cone to a metric cone at its tip is the cone itself.

Structure theory: the state of the art

Definition (k-regular set)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \le k \le N$ let

$$\mathcal{R}_k := \left\{ x \in X : \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathsf{d}_{eucl}, c_k \mathscr{L}^k, 0^k \right) \right\} \right\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k \right) = 0.$$

Furthermore, for any $1 \le k \le N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$, for some density $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner \mathcal{R}_k)$.

$(\mathfrak{m}, k)$-rectifiable means that, up to an $\mathfrak{m}$-negligible set, we can cover it with a countable union of biLipschitz images of subsets of $\mathbb{R}^k$.

Structure theory: the state of the art

Definition (k-regular set)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \le k \le N$ let

$$\mathcal{R}_k := \left\{ x \in X : \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathsf{d}_{eucl}, c_k \mathscr{L}^k, 0^k \right) \right\} \right\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k \right) = 0.$$

Furthermore, for any $1 \le k \le N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$, for some density $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner \mathcal{R}_k)$.

$(\mathfrak{m}, k)$-rectifiable means that, up to an $\mathfrak{m}$-negligible set, we can cover it with a countable union of biLipschitz images of subsets of $\mathbb{R}^k$.

Structure theory: the state of the art

Definition (k-regular set)

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \leq k \leq N$ let

$$\mathcal{R}_k := \left\{ x \in X : \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k \right) \right\} \right\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k \right) = 0.$$

Furthermore, for any $1 \leq k \leq N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$, for some density $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner \mathcal{R}_k)$.

$(\mathfrak{m}, k)$-rectifiable means that, up to an $\mathfrak{m}$-negligible set, we can cover it with a countable union of biLipschitz images of subsets of $\mathbb{R}^k$.

Structure theory: the state of the art

Definition (k-regular set)

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \leq k \leq N$ let

$$\mathcal{R}_k := \left\{x \in X : \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}) = \left\{\left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k\right)\right\}\right\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k\right) = 0.$$

Furthermore, for any $1 \leq k \leq N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$, for some density $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$.

$(\mathfrak{m}, k)$-rectifiable means that, up to an $\mathfrak{m}$-negligible set, we can cover it with a countable union of biLipschitz images of subsets of $\mathbb{R}^k$.

Structure theory: the state of the art

Definition (k-regular set)

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \leq k \leq N$ let

$$\mathcal{R}_k := \left\{ x \in X : \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k \right) \right\} \right\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k \right) = 0.$$

Furthermore, for any $1 \leq k \leq N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$, for some density $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$.

$(\mathfrak{m}, k)$-rectifiable means that, up to an $\mathfrak{m}$-negligible set, we can cover it with a countable union of biLipschitz images of subsets of $\mathbb{R}^k$.

Structure theory: the state of the art

Definition (k-regular set)

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \leq k \leq N$ let

$$\mathcal{R}_k := \left\{ x \in X : \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k \right) \right\} \right\}.$$

After [Mondino-Naber '14] and [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16], [Gigli-Pasqualetto '16] we have

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k \right) = 0.$$

Furthermore, for any $1 \leq k \leq N$ *the* k*-regular set* $\mathcal{R}_k$ *is* $(\mathfrak{m}, k)$*-rectifiable and* $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$*, for some density* $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$*.*

$(\mathfrak{m}, k)$-rectifiable means that, up to an $\mathfrak{m}$-negligible set, we can cover it with a countable union of biLipschitz images of subsets of $\mathbb{R}^k$.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K,N)$ spaces have constant dimension in the almost everywhere sense. This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K,N)$ spaces have constant dimension in the almost everywhere sense. This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Comments

Remark

In [Brué-S. 18] we proved that $\mathrm{RCD}(K, N)$ spaces have constant dimension in the almost everywhere sense.This was proved for Ricci limit spaces in [Colding-Naber '12].

Different approach needed with respect to the structure theory for Ricci limits:

- no notion of Hessian on RCD spaces at the time of [Mondino-Naber '14];
- failure of a key lemma of [Cheeger-Colding '97] on smooth weighted CD manifolds.

Key tools:

New almost splitting via excess theorem in [Mondino-Naber '14] for the rectifiable structure.

Use of the deep results by [De Philippis-Rindler '16] on the structure of solutions to linear PDEs for the absolute continuity of the reference measure.

Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on $\mathrm{RCD}(K, \infty)$ spaces has been developed. See also [Savaré '14] and [Sturm '15].

Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.
Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\mathrm{Hess} f|^2 \mathrm{dm} \leq C_{K,N} \left(\int_{B_2(x)} (\Delta f)^2 \mathrm{dm} + \inf_{m \in \mathbb{R}} \int_{B_2(x)} ||\nabla f|^2 - m| \mathrm{dm} \right) - K \int_{B_2(x)} |\nabla f|^2 .$$

Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on $\mathrm{RCD}(K,\infty)$ spaces has been developed.See also [Savaré '14] and [Sturm '15].

Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.
Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\mathrm{Hess} f|^2 \mathrm{dm} \leq C_{K,N} \left(\int_{B_2(x)} (\Delta f)^2 \mathrm{dm} + \inf_{m \in \mathbb{R}} \int_{B_2(x)} \left| |\nabla f|^2 - m \right| \mathrm{dm} \right) - K \int_{B_2(x)} |\nabla f|^2 .$$

Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on $\mathrm{RCD}(K,\infty)$ spaces has been developed.See also [Savaré '14] and [Sturm '15].

Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.
Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\mathrm{Hess} f|^2 \mathrm{dm} \le C_{K,N}\left(\int_{B_2(x)} (\Delta f)^2 \mathrm{dm} + \inf_{m\in\mathbb{R}} \int_{B_2(x)} \left||\nabla f|^2 - m\right| \mathrm{dm}\right) - K\int_{B_2(x)} |\nabla f|^2 .$$

Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on $\mathrm{RCD}(K,\infty)$ spaces has been developed.See also [Savaré '14] and [Sturm '15].

Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.
Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\mathrm{Hess} f|^2 \mathrm{d}\mathfrak{m} \leq C_{K,N} \left(\int_{B_2(x)} (\Delta f)^2 \mathrm{d}\mathfrak{m} + \inf_{m\in\mathbb{R}} \int_{B_2(x)} \left| |\nabla f|^2 - m \right| \mathrm{d}\mathfrak{m} \right) - K \int_{B_2(x)} |\nabla f|^2 .$$

Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on $\mathrm{RCD}(K,\infty)$ spaces has been developed.See also [Savaré '14] and [Sturm '15].

Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.
Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\mathrm{Hess} f|^2 \mathrm{dm} \leq C_{K,N} \left(\int_{B_2(x)} (\Delta f)^2 \mathrm{dm} + \inf_{m\in\mathbb{R}} \int_{B_2(x)} \left| |\nabla f|^2 - m \right| \mathrm{dm} \right) - K \int_{B_2(x)} |\nabla f|^2 .$$

Hessian on RCD spaces

In [Gigli '14] a second order differential calculus on $\mathrm{RCD}(K, \infty)$ spaces has been developed.See also [Savaré '14] and [Sturm '15].

Remark

The Hessian verifies many of the usual calculus rules.

Functions in the domain of the Laplacian have Hessian.
Moreover integrating Bochner's inequality against good cut-offs we get

$$\int_{B_1(x)} |\mathrm{Hess} f|^2 \mathrm{d}\mathfrak{m} \leq C_{K,N} \left(\int_{B_2(x)} (\Delta f)^2 \mathrm{d}\mathfrak{m} + \inf_{m \in \mathbb{R}} \int_{B_2(x)} \left| |\nabla f|^2 - m \right| \mathrm{d}\mathfrak{m} \right) - K \int_{B_2(x)} |\nabla f|^2.$$

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess}\, u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Bruè-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess}\, u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Bruè-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess}\, u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Brué-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess} u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Bruè-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess} u|^2 \mathrm{d}\mathfrak{m} \leq \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \leq \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Bruè-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess}\, u|^2 \mathrm{d}\mathfrak{m} \leq \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \leq \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

In [Bruè-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess}\, u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

δ-splitting maps and structure theory

In [Brué-Pasqualetto-S. 20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess} u|^2 \mathrm{d}\mathfrak{m} \leq \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \leq \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

δ-splitting maps and structure theory

In [Brué-Pasqualetto-S. '20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess} u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

δ-splitting maps and structure theory

In [Brué-Pasqualetto-S. 20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess} u|^2 \mathrm{d}\mathfrak{m} \le \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \le \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

δ-splitting maps and structure theory

In [Brué-Pasqualetto-S. 20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

Harmonic δ-splitting maps

Definition (δ-splitting map)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $B_r(x) \subset X$. We say that $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map provided:

a) u has harmonic and C_N-Lipschitz components;

b) $\frac{r^2}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\mathrm{Hess} u|^2 \mathrm{d}\mathfrak{m} \leq \delta$;

c) $\frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \mathrm{d}\mathfrak{m} \leq \delta$.

They have played a fundamental role in the theory of Ricci limits, as in [Cheeger-Colding '97], [Cheeger-Naber '15], [Cheeger-Jiang-Naber '18].

δ-splitting maps and structure theory

In [Brué-Pasqualetto-S. 20] we give new proofs of the structure theorems for $\mathrm{RCD}(K, N)$ spaces using δ-splitting maps to rectify, as in the Cheeger-Colding theory.

This tool allows to extend the theory to positive codimension.

The intuition

Suppose $X = \mathbb{R}^k \times Y$ and let $u := (x_1, \dots, x_k)$ be the canonical coordinates on the factor $\mathbb{R}^k$.

They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

Remark

Harmonic δ-splitting maps are harmonic approximations in the $W^{1,2}$-sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

The intuition

Suppose $X = \mathbb{R}^k \times Y$ and let $u := (x_1, \ldots, x_k)$ be the canonical coordinates on the factor $\mathbb{R}^k$.
They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

Remark

Harmonic δ-splitting maps are harmonic approximations in the $W^{1,2}$-sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

If $r^2|K| \leq \delta$ then condition b) in the definition of δ-splitting map is redundant.

The intuition

Suppose $X = \mathbb{R}^k \times Y$ and let $u := (x_1, \dots, x_k)$ be the canonical coordinates on the factor $\mathbb{R}^k$.
They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

Remark

Harmonic δ-splitting maps are harmonic approximations in the $W^{1,2}$-sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

If $r^2|K| \leq \delta$ then condition b) in the definition of δ-splitting map is redundant.

The intuition

Suppose $X = \mathbb{R}^k \times Y$ and let $u := (x_1, \ldots, x_k)$ be the canonical coordinates on the factor $\mathbb{R}^k$.
They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

Remark

Harmonic δ-splitting maps are harmonic approximations in the $W^{1,2}$-sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

Remark

If $r^2|K| \leq \delta$ then condition b) in the definition of δ-splitting map is redundant.

The intuition

Suppose $X = \mathbb{R}^k \times Y$ and let $u := (x_1, \dots, x_k)$ be the canonical coordinates on the factor $\mathbb{R}^k$.
They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

Remark

Harmonic δ-splitting maps are harmonic approximations in the $W^{1,2}$-sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

Remark

If $r^2|K| \leq \delta$ then condition b) in the definition of δ-splitting map is redundant.

The intuition

Suppose $X = \mathbb{R}^k \times Y$ and let $u := (x_1, \dots, x_k)$ be the canonical coordinates on the factor $\mathbb{R}^k$.
They are harmonic, with vanishing Hessians and (pointwise) orthogonal gradients.

Remark

Harmonic δ-splitting maps are harmonic approximations in the $W^{1,2}$-sense of the canonical Euclidean coordinates.

They are obtained relying on the compactness and stability results of [Gigli-Mondino-Savaré '15], [Ambrosio-Honda '17], [Ambrosio-Honda '18].

Remark

If $r^2|K| \leq \delta$ then condition b) in the definition of δ-splitting map is redundant.

δ-splitting implies ε-isometry

If at a certain location a map is δ-splitting at all sufficiently small scales, then any tangent is close to a split space.

Theorem

Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N,\varepsilon} > 0$ such that the following holds.
If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, and there exists a map $u : B_r(x) \to \mathbb{R}^k$ such that $u : B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for all $s < r$, then for any $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$

$$\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z)) \Big) \leq \varepsilon,$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

δ-splitting implies ε-isometry

If at a certain location a map is δ-splitting at all sufficiently small scales, then any tangent is close to a split space.

Theorem

Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N,\varepsilon} > 0$ such that the following holds.
If $(X, \mathrm{d}, \mathrm{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, and there exists a map $u\colon B_r(x) \to \mathbb{R}^k$ such that $u\colon B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for all $s < r$, then for any $(Y, \varrho, \mathrm{n}, y) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathrm{m})$

$$\mathrm{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathrm{n}, y), (\mathbb{R}^k \times Z, \mathrm{d}_{\mathrm{Eucl}} \times \mathrm{d}_Z, \mathcal{L}^k \otimes \mathrm{m}_Z, (0^k, z)) \Big) \le \varepsilon,$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathrm{d}_Z, \mathrm{m}_Z, z)$.

δ-splitting implies ε-isometry

If at a certain location a map is δ-splitting at all sufficiently small scales, then any tangent is close to a split space.

Theorem

Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N,\varepsilon} > 0$ such that the following holds.

If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, and there exists a map $u\colon B_r(x) \to \mathbb{R}^k$ such that $u\colon B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for all $s < r$, then for any $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$

$$\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z))\Big) \le \varepsilon,$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

δ-splitting implies ε-isometry

If at a certain location a map is δ-splitting at all sufficiently small scales, then any tangent is close to a split space.

Theorem

Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N,\varepsilon} > 0$ such that the following holds.

If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, and there exists a map $u\colon B_r(x) \to \mathbb{R}^k$ such that $u\colon B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for all $s < r$, then for any $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$

$$\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z))\Big) \le \varepsilon,$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

δ-splitting implies ε-isometry

If at a certain location a map is δ-splitting at all sufficiently small scales, then any tangent is close to a split space.

Theorem

Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N,\varepsilon} > 0$ such that the following holds.
If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, and there exists a map $u\colon B_r(x) \to \mathbb{R}^k$ such that $u\colon B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for all $s < r$, then for any $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$

$$\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), \big(\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z)\big)\Big) \leq \varepsilon,$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

ε-close yields δ-splitting

If a space is ε-close to a split space with Euclidean factor $\mathbb{R}^k$, then there is a (k, δ)-splitting map.

Theorem

Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that the following holds.
If $(X, \mathrm{d}, \mathrm{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, $r > 0$ with $r^2|K| \leq \varepsilon$, and there is an $\mathrm{RCD}(0, N-k)$ space $(Z, \mathrm{d}_Z, \mathrm{m}_Z, z)$ such that

$$\mathrm{d}_{\mathrm{pmGH}}\Big((X, \mathrm{d}/r, \mathrm{m}_x^r, x), (\mathbb{R}^k \times Z, \mathrm{d}_{\mathrm{Eucl}} \times \mathrm{d}_Z, \mathcal{L}^k \otimes \mathrm{m}_Z, (0^k, z)) \Big) \leq \varepsilon,$$

then there exists a δ-splitting map $u : B_{5r}(x) \to \mathbb{R}^k$.

ε-close yields δ-splitting

If a space is ε-close to a split space with Euclidean factor $\mathbb{R}^k$, then there is a (k,δ)-splitting map.

Theorem

Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that the following holds.
If $(X, \mathrm{d}, \mathrm{m})$ is an $\mathrm{RCD}(K,N)$ space, $x \in X$, $r > 0$ with $r^2|K| \leq \varepsilon$, and there is an $\mathrm{RCD}(0, N-k)$ space $(Z, \mathrm{d}_Z, \mathrm{m}_Z, z)$ such that

$$\mathrm{d}_{\mathrm{pmGH}}\Big((X, \mathrm{d}/r, \mathrm{m}_x^r, x), (\mathbb{R}^k \times Z, \mathrm{d}_{\mathrm{Eucl}} \times \mathrm{d}_Z, \mathcal{L}^k \otimes \mathrm{m}_Z, (0^k, z))\Big) \leq \varepsilon,$$

then there exists a δ-splitting map $u : B_{5r}(x) \to \mathbb{R}^k$.

ε-close yields δ-splitting

If a space is ε-close to a split space with Euclidean factor $\mathbb{R}^k$, then there is a (k, δ)-splitting map.

Theorem

Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that the following holds.
If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, $r > 0$ with $r^2|K| \le \varepsilon$, and there is an $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/r, \mathfrak{m}_x^r, x), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z)) \Big) \le \varepsilon,$$

then there exists a δ-splitting map $u \colon B_{5r}(x) \to \mathbb{R}^k$.

ε-close yields δ-splitting

If a space is ε-close to a split space with Euclidean factor $\mathbb{R}^k$, then there is a (k, δ)-splitting map.

Theorem

Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that the following holds.

If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, $r > 0$ with $r^2|K| \leq \varepsilon$, and there is an $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/r, \mathfrak{m}_x^r, x), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z)) \Big) \leq \varepsilon,$$

then there exists a δ-splitting map $u \colon B_{5r}(x) \to \mathbb{R}^k$.

ε-close yields δ-splitting

If a space is ε-close to a split space with Euclidean factor $\mathbb{R}^k$, then there is a (k, δ)-splitting map.

Theorem

Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that the following holds.
If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, $x \in X$, $r > 0$ with $r^2|K| \leq \varepsilon$, and there is an $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/r, \mathfrak{m}_x^r, x), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z)) \Big) \leq \varepsilon,$$

then there exists a δ-splitting map $u \colon B_{5r}(x) \to \mathbb{R}^k$.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then for* $\mathfrak{m}$*-a.e.* $x \in X$ *there exists* $1 \le k \le N$ *such that*

$$(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then for* $\mathfrak{m}$*-a.e.* $x \in X$ *there exists* $1 \le k \le N$ *such that*

$$(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then for* $\mathfrak{m}$*-a.e.* $x \in X$ *there exists* $1 \leq k \leq N$ *such that*

$$\left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k\right) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Remark

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. Then for $\mathfrak{m}$-a.e. $x \in X$ there exists $1 \le k \le N$ such that

$$\left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k\right) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Remark

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then for* $\mathfrak{m}$*-a.e.* $x \in X$ *there exists* $1 \leq k \leq N$ *such that*

$$\left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k\right) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Remark

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then for* $\mathfrak{m}$*-a.e.* $x \in X$ *there exists* $1 \le k \le N$ *such that*

$$\left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k\right) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Remark

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Existence of regular tangents

The starting point is the following

Theorem ([Gigli-Mondino-Rajala '13])

Let $(X, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s.. Then for* $\mathfrak{m}$*-a.e.* $x \in X$ *there exists* $1 \leq k \leq N$ *such that*

$$\left(\mathbb{R}^k, \mathrm{d}_{eucl}, c_k \mathscr{L}^k, 0^k\right) \in \mathrm{Tan}_x(X, \mathrm{d}, \mathfrak{m}).$$

Remark

Proved by iterative application of the splitting theorem, relying on the principle that tangents of tangents are tangents [Preiss '87], [Le Donne '11].

We also obtain an independent proof of the existence of regular tangents via a different strategy.

Propagation of regularity

Why using harmonic δ-splitting maps

Remark

In the study of singular sets on non collapsed Ricci limits, harmonic δ-splitting maps verify much better estimates than general δ-splitting maps, see [Cheeger-Jiang-Naber '18].

Also on general $\mathrm{RCD}(K, N)$ spaces harmonic δ-splitting maps turn to be useful.

Remark

In [Brué-Pasqualetto-S. '19] they are used to prove rectifiability of reduced boundaries for sets of finite perimeter.

Why using harmonic δ-splitting maps

Remark

In the study of singular sets on non collapsed Ricci limits, harmonic δ-splitting maps verify much better estimates than general δ-splitting maps, see [Cheeger-Jiang-Naber '18].

Also on general $\mathrm{RCD}(K, N)$ spaces harmonic δ-splitting maps turn to be useful.

Remark

In [Brué-Pasqualetto-S. '19] they are used to prove rectifiability of reduced boundaries for sets of finite perimeter.

Why using harmonic δ-splitting maps

Remark

In the study of singular sets on non collapsed Ricci limits, harmonic δ-splitting maps verify much better estimates than general δ-splitting maps, see [Cheeger-Jiang-Naber '18].

Also on general $\mathrm{RCD}(K, N)$ spaces harmonic δ-splitting maps turn to be useful.

Remark

In [Brué-Pasqualetto-S. '19] they are used to prove rectifiability of reduced boundaries for sets of finite perimeter.

Possible directions

Infinite dimensional case
Non linear case

Thank you for your attention!