The metric measure boundary of non collapsed RCD spaces

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Outline

Taylor expanding the volume of balls

On a smooth Riemannian manifold (X, d_g) endowed with its Riemannian volume measure $\mathcal{H}^n$, for any interior point x it holds

$$\mathcal{V}_r(x) := 1 - \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n} = \frac{\mathrm{Scal}(x)}{6(n+2)} r^2 + O(r^4), \quad r \downarrow 0.$$

For a boundary point $y \in \partial X$ it holds

$$\mathcal{V}_r(y) = \frac{1}{2} + o(1), \quad r \downarrow 0.$$

Lemma

On smooth Riemannian manifolds with boundary, the measures $r^{-1}\mathcal{V}_r\mathcal{H}^n$ weakly converge to $c_n\mathcal{H}^{n-1} \llcorner \partial X$ as $r \downarrow 0$.

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Remarks

If $(X, \mathrm{d}, \mathcal{H}^n)$ is an $\mathrm{RCD}(K, n)$ space, then by Bishop-Gromov

$$\mathcal{V}_r(x) \geq C_{K,n} r^2 , \quad 0 < r < 1 .$$

However the pointwise (both-sided) second order Taylor expansion can fail:

- at singular points $x \in X$

$$\lim_{r \to 0} \mathcal{V}_r(x) = 1 - \theta_n(x) > 0 ;$$

- there might be regular points $x \in X$ such that for some $0 < \alpha < 2$

$$\mathcal{V}_r(x) \sim r^{\alpha} , \quad r \downarrow 0 .$$

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Metric measure boundary

Definition (Kapovitch-Lytchak-Petrunin '17)

A metric measure space $(X, \mathrm{d}, \mathcal{H}^n)$ has finite metric measure boundary if the measures

$$\mu_r(\mathrm{d}x) := \frac{1}{r}\mathcal{V}_r(x)\mathcal{H}^n(\mathrm{d}x)$$

are locally uniformly bounded for $0 < r < 1$. If $\mu_r \to \nu$ weakly as $r \downarrow 0$, then ν is called metric measure boundary of $(X, \mathrm{d}, \mathcal{H}^n)$.

Any Alexandrov space with empty boundary has vanishing metric measure boundary.

In [KLP17] the conjecture was established in two cases:

- two dimensional Alexandrov surfaces;
- boundaries of convex bodies (with the induced metric).

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Definition (De Philippis-Gigli '18)

The boundary of an $\mathrm{RCD}(K, n)$ space $(X, \mathsf{d}, \mathcal{H}^n)$ is the (topological) closure of the set of points with a flat half-space in the tangent cone:

$$\partial X := \overline{\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2}} .$$

In [Bruè-Naber-S. '20] we proved that

- $(X, \mathsf{d}, \mathcal{H}^n)$ has empty boundary according to [DPG18] iff it has empty boundary according to [Kapovitch-Mondino '19];
- ∂X is $(n-1)$-rectifiable with locally finite $\mathcal{H}^{n-1}$ measure;
- at $\mathcal{H}^{n-1}$-a.e. $x \in \partial X$ the tangent cone is unique and isometric to a half-space.

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Metric measure boundary on RCD spaces

Theorem (Bruè-Mondino-S. '22)

There exists a constant $C(K,n,v_0)>0$ such that if $(X,\mathsf{d},\mathcal{H}^n)$ is an $\mathrm{RCD}(K,n)$ space, $p\in X$ verifies $\mathcal{H}^n(B_1(p))>v_0$ and $B_2(p)\cap\partial X=\emptyset$, then

$$|\mu_r|(B_1(p))\le C(K,n,v_0)\,,\quad \text{for any } 0<r<1$$

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$$|\mu_r|(B_1(p))\to 0\,,\quad \text{as } r\to 0\,.$$

In particular, if $(X,\mathsf{d},\mathcal{H}^n)$ has empty boundary, then it has vanishing metric measure boundary.

The statement is completely non trivial also for smooth manifolds, because the constant depends only on K, n and v_0.

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Consequences for Alexandrov spaces

Question [Perelman-Petrunin '96]

Do infinite geodesics exist on any Alexandrov space with empty boundary?

Theorem (Kapovitch-Lytchak-Petrunin '17)

If the m.m.b. of an Alexandrov space (X, d) vanishes, then a.e. direction in TX is the starting direction of an infinite geodesic. Moreover the geodesic flow preserves the Liouville measure on TX.

By [Petrunin '09] and [Zhang-Zhu '09] Alexandrov spaces are RCD.

Corollary (KLP17+BMS22)

The conclusion above holds on any Alexandrov space with empty boundary. Hence "many" infinite geodesics exist.

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Reaching the boundary

Theorem (Brué-Mondino-S. '22)

Let $(X,\mathrm{d},\mathcal{H}^n)$ be an $\mathrm{RCD}(K,n)$ space with boundary ∂X such that its doubling along the boundary is $\mathrm{RCD}(K,n)$. Then

$$\mu_r(\mathrm{d}\,x) := \frac{1}{r}\mathcal{V}_r(x)\mathcal{H}^n(\mathrm{d}\,x) \to c_n\mathcal{H}^{n-1}\llcorner\partial X\,, \quad \text{as } r \downarrow 0\,.$$

The additional assumption seems technical. It is satisfied for Alexandrov spaces by [Perelman '91] and non collapsed Ricci limit spaces with boundary by [Schlichting '12] (and "stability").

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Proof divided in two parts:

i) uniform mass estimates of the form $r^{-1}|\mu_r| \le f\mathcal{H}^n$ for some $L^1(\mathcal{H}^n)$ function $f : X \to [0, \infty]$;

ii) the density w.r.t. $\mathcal{H}^n$ of any weak limit $\nu = \lim r^{-1}\mu_r$ is 0 a.e.

Key ingredients for (i):

a) quantitative volume convergence theorem for almost Euclidean balls via splitting functions;

b) control of splitting maps away from sets of small $(n-1)$-dimensional content and iteration;

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Key ingredient for (ii): construction of splitting functions with “vanishing Hessian” at a.e. point.

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By [Colding '97], [Cheeger-Colding '97] and [De Philippis-Gigli '18], the volume $\mathcal{H}^n$ is continuous under GH convergence for spaces with $\dim \le n$ and $\mathrm{Ric} \ge -(n-1)$.

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Key to such a statement is the continuity when the limit space is $\mathbb{R}^n$.

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The following quantitative volume convergence via splitting maps holds

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In order to prove that the m.m.b. vanishes (when $\partial X = \emptyset$) in [BMS22] we construct at a.e. point "splitting" functions with "vanishing Hessians" and orthonormal gradients.

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Comparison with Alexandrov techniques

In [Kapovitch-Lytchak-Petrunin '17] it is shown that Alexandrov spaces have finite metric measure boundary.

Remark

The proof uses (smoothings of) distance coordinates in place of harmonic splitting functions.

Key differences between distance charts and harmonic charts (in Alexandrov geometry):

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Higher order expansion

Definition (Kapovitch-Lytchak-Petrunin '17)

A metric measure space $(X, \mathrm{d}, \mathcal{H}^n)$ has finite metric measure curvature if the measures

$$\sigma_r(\mathrm{d}x) := \frac{1}{r^2}\left(1 - \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n}\right)\mathcal{H}^n(\mathrm{d}x)$$

are locally uniformly bounded for $0 < r < 1$. If $\sigma_r \to \sigma$ weakly as $r \downarrow 0$, then σ is called metric measure curvature of $(X, \mathrm{d}, \mathcal{H}^n)$.

Smooth Riemannian manifolds with $\partial X = \emptyset$, $\mathrm{Ric} \geq -(n-1)$, $\mathcal{H}^n(B_1(p)) \geq v_0$ verify

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Integral scalar curvature bounds

Theorem (Petrunin '08)

There exists a constant $C(n) > 0$ such that for any smooth Riemannian manifold (M^n, g) with sectional curvature bounded below by -1 it holds

$$\frac{1}{\mathrm{vol}(B_1(p))} \int_{B_1(p)} |\mathrm{Scal}| \, \mathrm{dvol} \leq C(n) \, .$$

An analogous integral bound holds if the lower sectional curvature bound is replaced by a lower Ricci curvature bound, if needed, add a "non collapsing" assumption.

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If the conjectural bound on the metric measure curvature holds, then Yau's conjecture follows taking the limit as $r \downarrow 0$.

The converse implication seems more subtle.

The a priori bounds on the metric measure curvature are not known for smooth manifolds with sectional bounded below.

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Reaching the boundary

In order to fully establish the equivalence between m.m.b. and boundary measure it seems relevant to address the following:

Conjecture [Kapovitch-Ketterer-Sturm '20]

The doubling along the boundary of an $\mathrm{RCD}(K,n)$ space $(X,\mathsf{d},\mathcal{H}^n)$ is $\mathrm{RCD}(K,n)$.

A key step for the above seems to be the following:

Let $(X,\mathsf{d},\mathcal{H}^n)$ be $\mathrm{RCD}(K,n)$ with boundary ∂X. Then

$$\Delta \mathsf{d}_{\partial X} \le -K \mathsf{d}_{\partial X} \quad \text{on } X \setminus \partial X$$

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