

# The metric measure boundary of non collapsed RCD spaces

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# Outline

- 1 Introduction and motivations
- 2 Main results and consequences
- 3 Outline of the proof
- 4 Open questions

# Taylor expanding the volume of balls

On a smooth Riemannian manifold  $(X, d_g)$  endowed with its Riemannian volume measure  $\mathcal{H}^n$ , for any interior point  $x$  it holds

$$\mathcal{V}_r(x) := 1 - \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n} = \frac{\text{Scal}(x)}{6(n+2)} r^2 + O(r^4), \quad r \downarrow 0.$$

For a boundary point  $y \in \partial X$  it holds

$$\mathcal{V}_r(y) = \frac{1}{2} + o(1), \quad r \downarrow 0.$$

QUESTION

Draw a smooth Riemannian manifold with boundary. The measure of  $\mathcal{V}_r(x)$  and  $\mathcal{V}_r(y)$  for  $x \in X$  and  $y \in \partial X$  as  $r \downarrow 0$ .

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Lemma

*On smooth Riemannian manifolds with boundary, the measures  $r^{-1} \mathcal{V}_r \mathcal{H}^n$  weakly converge to  $c_r \mathcal{H}^{n-1} \llcorner \partial X$  as  $r \downarrow 0$ .*

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## Lemma

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# Remarks

If  $(X, d, \mathcal{H}^n)$  is an  $\text{RCD}(K, n)$  space, then by **Bishop-Gromov**

$$\mathcal{V}_r(x) \geq C_{K,n} r^2, \quad 0 < r < 1.$$

However the **pointwise** (both-sided) second order Taylor expansion can **fail**:

- at singular points  $x \in X$

$$\lim_{r \rightarrow 0} \mathcal{V}_r(x) = 1 - \theta_n(x) > 0;$$

but this might be false for some  $x \in X$  with  $\dim_x x < n$  and  $\theta_n(x) = 0$ .

$$\mathcal{V}_r(x) \sim r^{\dim_x x} \quad (x \in X)$$

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# Metric measure boundary

Definition (Kapovich-Lylichak-Petrinin '17)

A metric measure space  $(X, d, \mathcal{H}^n)$  has finite metric measure boundary if the measures

$$\mu_r(dx) := \frac{1}{r} \nu_r(x) \mathcal{H}^n(dx)$$

are locally uniformly bounded for  $0 < r < 1$ . If  $\mu_r \rightarrow \nu$  weakly as  $r \downarrow 0$ , then  $\nu$  is called metric measure boundary of  $(X, d, \mathcal{H}^n)$ .

Conjecture (Kapovich-Lylichak-Petrinin '17)

Any Alexandrov space with convex boundary has vanishing metric measure boundary.

In [KLP17] the conjecture was established in two cases:

- in the 2-dimensional case (see below)
- in the case of convex bodies (with the induced metric).

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## Conjecture (Kapovitch-Lytchak-Petrinin '17)

Any Alexandrov space with empty boundary has vanishing metric measure boundary.

In [KLP17] the conjecture was established in two cases:

- Alexandrov spaces with empty boundary and bounded diameter
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## Conjecture [Kapovitch-Lytchak-Petrinin '17]

Any **Alexandrov** space with **empty boundary** has **vanishing** metric measure boundary.

In [KLP17] the conjecture was established in two cases:

- two dimensional Alexandrov surfaces;
- Alexandrov spaces with bounded diameter.

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- **boundaries** of **convex bodies** (with the induced metric).

# Boundaries of RCD spaces

## Definition (De Philippis-Gigli '18)

The boundary of an RCD( $K, n$ ) space  $(X, d, \mathcal{H}^n)$  is the (topological) closure of the set of points with a flat half-space in the tangent cone:

$$\partial X := \overline{S^{n-1} \setminus S^{n-2}}.$$

In [Bruè-Naber-S. '20] we proved that

$(X, d, \mathcal{H}^n)$  has empty boundary according to [De Philippis-Gigli '18] if and only if it has empty boundary according to [Bruè-Naber-S. '20].

→ [Bruè-Naber-S. '20] →  $(X, d, \mathcal{H}^n)$  with locally finite  $\mathcal{H}^{n-1}$  measure.

→ at  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial X$  the tangent cone is unique and isomorphic to  $\mathbb{R}^n$ .

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In [Bruè-Naber-S. '20] we proved that

- $(X, d, \mathcal{H}^n)$  has empty boundary according to [DPG18] iff it has empty boundary according to [Kapovitch-Mondino '19];
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# Metric measure boundary on RCD spaces

Theorem (Bruè-Mondino-S. '22)

There exists a constant  $C(K, n, \nu_0) > 0$  such that if  $(X, d, \mathcal{H}^n)$  is an  $\text{RCD}(K, n)$  space,  $p \in X$  verifies  $\mathcal{H}^n(B_1(p)) > \nu_0$  and  $B_2(p) \cap \partial X = \emptyset$ , then

$$|\mu_r|(B_r(p)) \leq C(K, n, \nu_0), \quad \text{for any } 0 < r < 1$$

and

$$|\mu_r|(B_r(p)) \rightarrow 0, \quad \text{as } r \rightarrow 0.$$

In particular, if  $(X, d, \mathcal{H}^n)$  has empty boundary, then it has vanishing metric measure boundary.

The question is naturally raised: what does it mean to have a metric measure boundary? The answer is essentially the same as in the Euclidean case.



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Remark

The statement is completely non trivial also for smooth manifolds, because the constant depends only on  $K$ ,  $n$  and  $v_0$ .

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## Remark

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# Consequences for Alexandrov spaces

## Question [Perelman-Petrinin '96]

Do infinite geodesics exist on any Alexandrov space with empty boundary?

## Answer [Mogavich-Lyngbyh-Petrinin '17]

If the multiplicity of an Alexandrov space  $(X, d)$  vanishes, then  $x \in \text{int}(X)$  is the starting direction of an infinite geodesic. Moreover the geodesic flow preserves the Liouville measure on  $T_x X$ .

By [Petrinin '09] and [Zhang-Zhu '09] Alexandrov spaces are RCD.

## Conjecture [LPSZ+SW22]

The conclusion above holds on any Alexandrov space with empty boundary, hence there are no infinite geodesics.

# Consequences for Alexandrov spaces

## Question [Perelman-Petrinin '96]

Do **infinite geodesics** exist on any Alexandrov space with empty boundary?

## Theorem (Kapovitch-Lytchak-Petrinin '17)

*If the m.m.b. of an Alexandrov space  $(X, d)$  vanishes, then a.e. direction in  $TX$  is the starting direction of an infinite geodesic. Moreover the geodesic flow preserves the Liouville measure on  $TX$ .*

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## Conjecture [Lytchak-Petrinin '17]

*For m.m.b. spaces above holds on any  $d$ -space with empty boundary.*

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## Corollary [KLP17+BMS22]

*The conclusion above holds on any Alexandrov space with empty boundary. Hence "many" infinite geodesics exist.*

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# Reaching the boundary

Theorem (Brué-Mondino-S. '22)

Let  $(X, d, \mathcal{H}^n)$  be an  $\text{RCD}(K, n)$  space with boundary  $\partial X$  such that its doubling along the boundary is  $\text{RCD}(K, n)$ . Then

$$\mu_r(\mathrm{d}x) := \frac{1}{r} \nu_r(x) \mathcal{H}^n(\mathrm{d}x) \rightarrow c_n \mathcal{H}^{n-1} \llcorner \partial X, \quad \text{as } r \downarrow 0.$$

The additional assumption seems technical. It is satisfied for instance for spaces bounded by a smooth manifold and for spaces with  $\text{RCD}(K, n)$  boundary with "boundary stability".

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### Remark

The additional assumption seems technical. It is satisfied for Alexandrov spaces by [Fardman '91] and non-collapsed Ricci limit spaces with boundary by [Schlichting '12] (and "stability").

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## Remark

The additional assumption seems technical. It is satisfied for *Alexandrov spaces* by [Perelman '91] and *non collapsed Ricci limit spaces* with boundary by [Schlichting '12] (and “stability”).

# Proof outline

Proof divided in two parts:

- (i) uniform mass estimates of the form  $r^{-1}|\mu_r| \leq fH^n$  for some  $L^1(H^n)$  function  $f: X \rightarrow [0, \infty]$ ;
- (ii) the density w.r.t.  $H^n$  of any weak limit  $\mu \in \text{weak}^* \mu_r$  is 0 a.e.

Key ingredients for (i):

- (1) splitting maps (see [Colding-Preiss] or [Colding-DeSilva])
- (2) control of splitting maps away from sets of small  $(n-1)$ -dimensional Hausdorff and Lebesgue measure
- (3) control of  $L^1$  estimates (Colding-DeSilva)

Key ingredient for (ii): construction of splitting functions with “vanishing Hessian” at a.e. point.

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Proof divided in two parts:

- (i) uniform mass estimates of the form  $r^{-1}|\mu_r| \leq f\mathcal{H}^n$  for some  $L^1(\mathcal{H}^n)$  function  $f: X \rightarrow [0, \infty]$ ;
- (ii) the density w.r.t.  $\mathcal{H}^n$  of any weak limit  $\nu = \lim r^{-1}\mu_r$  is 0 a.e. .

Key ingredients for (i):

- 1) quantitative isoperimetric inequalities (Federer-Eisenberg data)
- 2) control of splitting maps away from sets of small  $(n-1)$ -dimensional perimeter and density
- 3) control of  $\mathcal{H}^n$ -measures along  $\mathbb{R}^n$ -directions

Key ingredient for (ii): construction of splitting functions with “vanishing Hessian” at a.e. point.

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- 1) **splitting maps** (Colding-De Lellis, Colding-De Lellis-Preiss, De Lellis-Preiss)
- 2) **control of splitting maps away from sets of small  $(n-1)$ -dimensional Hausdorff measure**
- 3) **control of the mass of the boundary of the splitting map**

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- 1) quantitative volume convergence theorem for almost Euclidean balls via splitting functions;
- 2) control of splitting maps away from sets of small  $\mathcal{H}^n$ -dimensional measure and large perimeter;
- 3) control of the volume of the boundary via the RCD condition.

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Key ingredients for (i):

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- c) control of the volume of the boundary of the balls.

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- a) **quantitative volume convergence** theorem for almost Euclidean balls via **splitting functions**;
- b) control of splitting maps away from sets of small  $(n - 1)$ -dimensional content and iteration;
- c) series of **quantitative covering arguments**.

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# Harmonic coordinates

In [Bruè-Naber-S. '20] extending the previous [Cheeger-Colding '96], [Cheeger-Naber '14], [Cheeger-Jiang-Naber '18] we proved:

## Theorem

*For any  $0 < \alpha < 1$  and  $\delta > 0$  there exists  $\varepsilon(\alpha, \delta, n) > 0$  such that if  $(X, d, \mathcal{H}^n)$  is an  $\text{RCD}(-\varepsilon, n)$  space and  $d_{\text{GH}}(B_2(p), B_2(0^n)) < \varepsilon$ , then there exists a harmonic map  $u : B_{3/2}(p) \rightarrow \mathbb{R}^n$  such that*

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- $$\sum_{i,j} \frac{1}{\mathcal{H}^n(B_1(p))} \int_{B_1(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \leq \delta;$$

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# Quantitative volume convergence via splitting maps

By [Colding '97], [Cheeger-Colding '97] and [De Philippis-Gigli '18], the volume  $\mathcal{H}^n$  is continuous under GH convergence for spaces with  $\dim \leq n$  and  $\text{Ric} \geq -(n-1)$ .

Remark

Key to such a statement is the continuity when the limit space is  $\mathbb{R}^n$ .

Proposition (Colding-Cheeger, 1997)

The total volume of a sequence of  $n$ -dimensional manifolds with bounded diameter and Ricci curvature bounded from below converges to the volume of the limit space.

$$\lim_{i \rightarrow \infty} \int_{M_i} \frac{dV}{d\mathcal{H}^n} = \int_{M_\infty} \frac{dV}{d\mathcal{H}^n} + \int_{M_\infty} \nu_{\text{split}} \nu_{\text{split}}^{-1} d\mathcal{H}^n$$

Colding and CMC proved that the limit space is a metric measure space with Ricci curvature bounded from below. This statement is very weak as it only controls the volume.



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Theorem (Brüé-Mondino S. '22)

The following quantitative volume convergence via splitting maps holds:

$$\left| 1 - \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n} \right| \leq C_n \left( \rho + \int_0^r \frac{\gamma}{t^p} \int_{B_t(x)} |\nabla u \cdot \nabla u - \kappa_t| d\mathcal{H}^n \frac{dt}{t} \right)$$

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It is understood at least from [Cheeger-Naber '14] (see also [Ding '01], [Petrulin '03] and [De Philippis-Zimbron '19]) that harmonic maps degenerate "at singular points". They might degenerate also at some regular points.

An argument originally due to [Cheeger-Colding '00] shows non degeneration away from sets of small codimension  $(2 - \eta)$ -content, for any  $0 < \eta < 2$ .

There is  $\varepsilon \in \mathcal{D}_\eta(\eta)$  such that

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- $E \subset \cup B_{r_i}(x_i)$ ,  $\sum_i r_i^{n-1} \leq C_n \delta$ ;
- for any  $x \in B_1(p) \setminus E$  it holds

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# Perturbations and vanishing Hessians

In order to prove that the m.m.b. vanishes (when  $\partial X = \emptyset$ ) in [BMS22] we construct at a.e. point “splitting” functions with “vanishing Hessians” and orthonormal gradients.

Lemma (Brué-Mondino-S. '22)

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# Comparison with Alexandrov techniques

In [Kapovitch-Lytchak-Petrinin '17] it is shown that Alexandrov spaces have finite metric measure boundary.

## Remark

The proof uses (smoothings of) distance coordinates in place of harmonic splitting functions.

Key differences between distance charts and harmonic charts (in Alexandrov geometry):

- distance coordinates have better "local" behavior than harmonic charts

- harmonic coordinates have better "global" behavior (e.g. Hölder in  $L^1$  vs Hölder measure)

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• distance coordinates are better suited to the study of the metric measure boundary;

• distance charts are easier to construct.



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# Higher order expansion

Definition (Kapovitch-Lytchak-Petrinin '17)

A metric measure space  $(X, d, \mathcal{H}^n)$  has finite metric measure curvature if the measures

$$\sigma_r(dx) := \frac{1}{r^2} \left( 1 - \frac{\mathcal{H}^n(B_1(x))}{\omega_n r^n} \right) \mathcal{H}^n(dx)$$

are locally uniformly bounded for  $0 < r < 1$ . If  $\sigma_r \rightarrow \sigma$  weakly as  $r \downarrow 0$ , then  $\sigma$  is called metric measure curvature of  $(X, d, \mathcal{H}^n)$ .

Sharp isoperimetric bounds with  $\delta X = 1$ ,  $\nu \geq -(n-1)$ ,  $\nu \in \mathbb{R}$  (in  $\mathbb{R}^n$ )

$$\frac{|dK(B)|}{2\pi^2} \leq C(n, \nu)$$

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Smooth Riemannian manifolds with  $\partial X = \emptyset$ ,  $\text{Ric} \geq -(n-1)$ ,  $\mathcal{H}^n(B_1(p)) \geq v_0$  verify

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# Integral scalar curvature bounds

## Theorem (Petrunin '08)

There exists a constant  $C(n) > 0$  such that for any smooth Riemannian manifold  $(M^n, g)$  with sectional curvature bounded below by  $-1$  it holds

$$\frac{1}{\text{vol}(B_1(\rho))} \int_{B_1(\rho)} |\text{Scal}| \, d\text{vol} \leq C(n).$$

An *integral scalar curvature bound* is a lower sectional curvature bound replaced by a lower Ricci curvature bound. It is called *integral* following [Petrunin 08].



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If the conjectural bound on the metric measure curvature holds, then Yau's conjecture follows taking the limit as  $r \downarrow 0$ .

The converse implication seems more subtle.

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# Reaching the boundary

In order to fully establish the **equivalence** between **m.m.b.** and **boundary measure** it seems relevant to address the following:

**Conjecture [Kapovitch-Ketterer-Sturm '20]**

The doubling along the boundary of an  $\text{RCD}(K, n)$  space  $(X, d, \mathcal{H}^n)$  is  $\text{RCD}(K, n)$ .

A key step for the above seems to be the following:

Let  $(X, d, \mathcal{H}^n) \in \text{RCD}(K, n)$  with boundary  $\partial X$ . Then

there exists  $\tilde{X} \supseteq X$  with  $\partial \tilde{X} = \partial X$  and  $\tilde{X} \in \text{RCD}(K, n)$ .

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Let  $\mathcal{H}^n \llcorner \partial X$  be the  $n$ -dimensional Hausdorff measure on  $\partial X$ .

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Thank you for your attention!