The metric measure boundary of non collapsed RCD spaces

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Introduction

I will discuss joint work with Elia Bruè and Andrea Mondino where we prove that the metric measure boundary vanishes for $\mathrm{RCD}(K, n)$ metric measure spaces $(X, \mathrm{d}, \mathcal{H}^n)$ with empty boundary.

In particular, the statement holds for Alexandrov spaces with curvature bounded below.

Combined with earlier work of Kapovitch-Lytchak-Petrunin, this answers a question raised by Perelman-Petrunin about the existence of infinite geodesics for Alexandrov spaces.

Outline

Brief introduction to RCD spaces

$\mathrm{RCD}(K,N)$ metric measure spaces $(X,\mathrm{d},\mathrm{m})$ are "Riemannian" spaces with Ricci bounded from below by $K\in\mathbb{R}$, dimension bounded above by $1\le N<\infty$.

Recall the Bochner identity:

$$\frac{1}{2}\Delta|\nabla f|^2=|\mathrm{Hess}\,f|^2_{\mathrm{HS}}\,+\,\nabla f\cdot\nabla\Delta f\,+\,\mathrm{Ric}(\nabla f,\nabla f)\,.$$

Definition

A m.m.s. $(X,\mathrm{d},\mathrm{m})$ is $\mathrm{RCD}(K,N)$ if:

- $W^{1,2}$ is a Hilbert space and functions with bounded gradient are Lipschitz;
- for sufficiently many test functions $f:X\to\mathbb{R}$,

$$\frac{1}{2}\Delta|\nabla f|^2\ge\frac{(\Delta f)^2}{N}+\nabla f\cdot\nabla\Delta f+K|\nabla f|^2\,.$$

Remarks

Contributions by several authors after [Sturm '04], [Lott-Villani '04] (for the CD Curvature-Dimension condition) and [Ambrosio-Gigli-Savaré '11] (for the additional Riemannian assumption).

The original approach (RCD iff (CD + $W^{1,2}$ is Hilbert)), is equivalent to the formulation in the previous slide thanks to [Ambrosio-Gigli-Savaré '12], [Erbar-Kuwada-Sturm '13], [Ambrosio-Mondino-Savaré '15] and [Cavalletti-Milman '16].

Alexandrov implies RCD

Remark

In the smooth Riemannian case a lower bound on the sectional curvature clearly implies a lower bound on the Ricci curvature.

Theorem (Petrunin '09, Zhang-Zhu '09)

If (X, d) is an n-dimensional Alexandrov space with curvature bounded below by $k \in \mathbb{R}$, then $(X, \mathrm{d}, \mathcal{H}^n)$ is an $\mathrm{RCD}(k(n-1), n)$ space.

The strategy is to show that any such space is a $\mathrm{CD}(k(n-1), n)$ space in the sense of Lott-Sturm-Villani.

Other examples

- Quotients of $\mathrm{RCD}(K, N)$ spaces with respect to isometric group actions are $\mathrm{RCD}(K, N)$ [Galaz-García-Kell-Mondino-Sosa '17].
- Ricci limit spaces $(M^n, g_i, \mathrm{vol}_i) \to (X, \mathrm{d}, \mathrm{m})$ with $\mathrm{Ric}_i \geq K$ are $\mathrm{RCD}(K, n)$ spaces.
- Cones and spherical suspensions over $\mathrm{RCD}(n-2, n-1)$ spaces are $\mathrm{RCD}(0, n)$ spaces [Ketterer '13, '15].

Some key properties

Theorem (Sturm '06)

The Bishop-Gromov inequality holds for $\mathrm{CD}(K, N)$ *spaces (hence it holds for* $\mathrm{RCD}(K, N)$ *spaces).*

Theorem (De Philippis-Gigli '18 after Cheeger-Colding '97 and Mondino-Naber '14)

Given any $\mathrm{RCD}(K, n)$ *space* $(X, \mathrm{d}, \mathcal{H}^n)$*, at* $\mathcal{H}^n$*-a.e. point the tangent cone is unique and isometric to* $\mathbb{R}^n$*.*

Self-improvement of Curvature-Dimension condition

A fundamental insight originally due to [Bakry '85] in the setting of Γ-calculus is that the Bochner inequality

$$\Delta \frac{1}{2}|\nabla f|^2 \geq \nabla f \cdot \nabla \Delta f + K|\nabla f|^2 \,,$$

for sufficiently many text functions f, self-improves to a Bochner inequality with Hessian term:

$$\Delta \frac{1}{2}|\nabla f|^2 \geq |\mathrm{Hess} f|^2_{\mathrm{HS}} + \nabla f \cdot \nabla \Delta f + K|\nabla f|^2 \,.$$

The principle was adapted to RCD spaces in [Gigli '14] building on top of the earlier [Savaré '13].

Taylor expanding the volume of balls

On a smooth Riemannian manifold (X, d_g) endowed with its Riemannian volume measure $\mathcal{H}^n$, for any interior point x it holds

$$\mathcal{V}_r(x) := 1 - \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n} = \frac{\mathrm{Scal}(x)}{6(n+2)} r^2 + O(r^4)\,, \quad r \downarrow 0\,.$$

For a boundary point $y \in \partial X$ it holds

$$\mathcal{V}_r(y) = \frac{1}{2} + o(1)\,, \quad r \downarrow 0\,.$$

Lemma

On smooth Riemannian manifolds with boundary, the measures $r^{-1}\mathcal{V}_r\mathcal{H}^n$ weakly converge to $c_n\mathcal{H}^{n-1} \llcorner \partial X$ as $r \downarrow 0$.

Remarks

If $(X, \mathrm{d}, \mathcal{H}^n)$ is an $\mathrm{RCD}(K, n)$ space, then by Bishop-Gromov

$$\mathcal{V}_r(x) \geq C_{K,n} r^2 \,, \quad 0 < r < 1 \,.$$

However the pointwise (both-sided) second order Taylor expansion can fail:

- at singular points $x \in X$

$$\lim_{r \to 0} \mathcal{V}_r(x) = 1 - \theta_n(x) > 0 \,;$$

- there might be regular points $x \in X$ such that for some $0 < \alpha < 2$

$$\mathcal{V}_r(x) \sim r^{\alpha} \,, \quad r \downarrow 0 \,.$$

Metric measure boundary

Definition (Kapovitch-Lytchak-Petrunin '17)

A metric measure space $(X, \mathrm{d}, \mathcal{H}^n)$ has finite metric measure boundary if the measures

$$\mu_r(\mathrm{d}x) := \frac{1}{r}\mathcal{V}_r(x)\mathcal{H}^n(\mathrm{d}x)$$

are locally uniformly bounded for $0 < r < 1$. If $\mu_r \to \nu$ weakly as $r \downarrow 0$, then ν is called metric measure boundary of $(X, \mathrm{d}, \mathcal{H}^n)$.

Conjecture [Kapovitch-Lytchak-Petrunin '17]

Any Alexandrov space with empty boundary has vanishing metric measure boundary.

In [KLP17] the conjecture was established in two cases:

- two dimensional Alexandrov surfaces;
- boundaries of convex bodies (with the induced metric).

Boundaries of RCD spaces

Definition (De Philippis-Gigli '18)

The boundary of an $\mathrm{RCD}(K, n)$ space $(X, \mathrm{d}, \mathcal{H}^n)$ is the (topological) closure of the set of points with a flat half-space in the tangent cone:

$$\partial X := \overline{\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2}}\,.$$

In [Bruè-Naber-S. '20] we proved that

- $(X, \mathrm{d}, \mathcal{H}^n)$ has empty boundary according to [DPG18] iff it has empty boundary according to [Kapovitch-Mondino '19];
- ∂X is $(n-1)$-rectifiable with locally finite $\mathcal{H}^{n-1}$ measure;
- at $\mathcal{H}^{n-1}$-a.e. $x \in \partial X$ the tangent cone is unique and isometric to a half-space.

Metric measure boundary on RCD spaces

Theorem (Bruè-Mondino-S. '22)

There exists a constant $C(K,n,v_0)>0$ such that if $(X,\mathrm{d},\mathcal{H}^n)$ is an $\mathrm{RCD}(K,n)$ space, $p\in X$ verifies $\mathcal{H}^n(B_1(p))>v_0$ and $B_2(p)\cap\partial X=\emptyset$, then

$$|\mu_r|(B_1(p))\le C(K,n,v_0)\,,\quad \text{for any } 0<r<1$$

and

$$|\mu_r|(B_1(p))\to 0\,,\quad \text{as } r\to 0\,.$$

In particular, if $(X,\mathrm{d},\mathcal{H}^n)$ has empty boundary, then it has vanishing metric measure boundary.

Remark

The statement is non trivial also for smooth manifolds, because the constant depends only on K, n and v_0.

Consequences for Alexandrov spaces

Question [Perelman-Petrunin '96]

Do infinite geodesics exist on any Alexandrov space with empty boundary?

Theorem (Kapovitch-Lytchak-Petrunin '17)

If the m.m.b. of an Alexandrov space (X, d) vanishes, then a.e. direction in TX is the starting direction of an infinite geodesic. Moreover the geodesic flow preserves the Liouville measure on TX.

By [Petrunin '09] and [Zhang-Zhu '09] Alexandrov spaces are RCD.

Corollary (KLP17+BMS22)

The conclusion above holds on any Alexandrov space with empty boundary. Hence "many" infinite geodesics exist.

Reaching the boundary

Theorem (Brué-Mondino-S. '22)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ *be an* $\mathrm{RCD}(K, n)$ *space with boundary* ∂X *such that its doubling along the boundary is* $\mathrm{RCD}(K, n)$*. Then*

$$\mu_r(\mathrm{d}\, x) := \frac{1}{r}\mathcal{V}_r(x)\mathcal{H}^n(\mathrm{d}\, x) \to c_n\mathcal{H}^{n-1} \llcorner \partial X\,, \quad \text{as } r \downarrow 0\,.$$

Remark

The additional assumption seems technical. It is satisfied for Alexandrov spaces by [Perelman '91] and non collapsed Ricci limit spaces with boundary by [Schlichting '12] (and "stability").

Proof outline

Proof divided in two parts:

i) uniform mass estimates of the form $|\mu_r| \le f\mathcal{H}^n$ for some $L^1(\mathcal{H}^n)$ function $f : X \to [0, \infty]$;

ii) the density w.r.t. $\mathcal{H}^n$ of any weak limit $\nu = \lim \mu_r$ is 0 a.e. .

Key ingredients for (i):

a) quantitative volume convergence theorem for almost Euclidean balls via splitting functions;

b) control of splitting maps away from sets of small $(n-1)$-dimensional content and iteration;

c) series of quantitative covering arguments.

Key ingredient for (ii): construction of splitting functions with "vanishing Hessian" at a.e. point.

Harmonic *coordinates*

In [Bruè-Naber-S. '20] extending the previous [Cheeger-Colding '96], [Cheeger-Naber '14], [Cheeger-Jiang-Naber '18] we proved:

Theorem

For any $0 < \alpha < 1$ *and* $\delta > 0$ *there exists* $\varepsilon(\alpha, \delta, n) > 0$ *such that if* $(X, \mathrm{d}, \mathcal{H}^n)$ *is an* $\mathrm{RCD}(-\varepsilon, n)$ *space and* $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \varepsilon$*, then there exists a harmonic map* $u : B_{3/2}(p) \to \mathbb{R}^n$ *such that*

- $|\nabla u| \leq 1 + \delta$ *(see also [Honda-Peng '22]);*
- $$\sum_{i,j} \frac{1}{\mathcal{H}^n(B_1(p))} \int_{B_1(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \leq \delta\,;$$
- $$\frac{1}{\mathcal{H}^n(B_1(p))} \int_{B_1(p)} |\mathrm{Hess}\, u|^2 \leq \delta\,;$$
- *u is a α-biHölder homeomorphism with its image, which contains* $B_{1/2}(0^n)$.

Quantitative volume convergence via splitting maps

By [Colding '97], [Cheeger-Colding '97] and [De Philippis-Gigli '18], the volume $\mathcal{H}^n$ is continuous under GH convergence for spaces with $\dim \leq n$ and $\mathrm{Ric} \geq -(n-1)$.

Remark

Key to such a statement is the continuity when the limit space is $\mathbb{R}^n$.

Theorem (Bruè-Mondino-S. '22)

The following quantitative volume convergence via splitting maps holds:

$$|1 - \frac{\mathcal{H}^n(B_r(x))}{\omega_n r^n}| \leq C_n \left(r^2 + \int_0^r \frac{1}{t^n} \int_{B_t(x)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \mathrm{d}\mathcal{H}^n \frac{\mathrm{d}t}{t} \right)$$

Remark

Following the proof in [Cheeger '01] one gets a worse exponent $\alpha(n) < 1$ at the RHS above. That estimate is too weak for controlling the m.m.b. .

Degeneration and non degeneration of splitting maps

Remark

It is understood at least from [Cheeger-Naber '14] (see also [Ding '01], [Petrunin '03] and [De Philippis-Zimbron '19]) that harmonic maps degenerate "at singular points". They might also degenerate at some regular points.

An argument originally due to [Cheeger-Colding '00] shows non degeneration away from sets of small codimension $(2-\eta)$-content, for any $0 < \eta < 2$.

Lemma

There is $E \subset B_1(p)$ such that:

- $E \subset \cup B_{r_i}(x_i)$, $\sum_i r_i^{n-1} \le C_n \delta$;
- *for any* $x \in B_1(p) \setminus E$ *it holds*

$$\sup_{0<r<1} \frac{1}{\mathcal{H}^n(B_r(x))} \int_{B_r(x)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \le C_n \sqrt{\delta}\,.$$

Comparison with Alexandrov techniques

In [Kapovitch-Lytchak-Petrunin '17] it is shown that Alexandrov spaces have finite metric measure boundary.

Remark

The proof uses (smoothings of) distance coordinates in place of harmonic splitting functions.

Key differences between distance charts and harmonic charts (in Alexandrov geometry):

- distance coordinates have better first order pointwise behaviour: they are biLipschitz;
- harmonic coordinates have better second order integral behaviour: Hessian in L^2 vs Hessian measure.

Higher order expansion

Definition (Kapovitch-Lytchak-Petrunin '17)

A metric measure space $(X, \mathrm{d}, \mathcal{H}^n)$ has finite metric measure curvature if the measures

$$\sigma_r(\mathrm{d}x) := \frac{1}{r^2}\left(1 - \frac{\mathcal{H}^n(B_1(x))}{\omega_n r^n}\right)\mathcal{H}^n(\mathrm{d}x)$$

are locally uniformly bounded for $0 < r < 1$. If $\sigma_r \to \sigma$ weakly as $r \downarrow 0$, then σ is called metric measure curvature of $(X, \mathrm{d}, \mathcal{H}^n)$.

Conjecture

Smooth Riemannian manifolds with $\partial X = \emptyset$, $\mathrm{Ric} \geq -(n-1)$, $\mathcal{H}^n(B_1(p)) > v_0$ verify

$$\frac{|\sigma_r|(B_1(p))}{\mathcal{H}^n(B_1(p))} \leq C(n, v_0)\,.$$

$\mathrm{RCD}(K, N)$ spaces with $\partial X = \emptyset$ $(X, \mathrm{d}, \mathcal{H}^n)$ have finite m.m. curvature.

Integral scalar curvature bounds

Theorem (Petrunin '08)

There exists a constant $C(n) > 0$ such that for any smooth Riemannian manifold (M^n, g) with sectional curvature bounded below by -1 it holds

$$\int_{B_1(p)} |\mathrm{Scal}|\, \mathrm{dvol} \leq C(n)\,.$$

Conjecture [Yau '90]

An analogous integral bound holds if the lower sectional curvature bound is replaced by a lower Ricci curvature bound. If needed, add a "non collapsing" assumption.

Remarks

Remark

If the conjectural bound on the metric measure curvature holds, then Yau's conjecture follows taking the limit as $r \downarrow 0$.

The converse implication seems more subtle.

Remark

The a priori bounds on the metric measure curvature are not known for smooth manifolds with sectional bounded below.

Remark [Kapovitch-Lytchak-Petrunin '17]

The metric measure curvature is not stable under GH convergence, even in dimension 2.

Reaching the boundary

In order to fully establish the equivalence between m.m.b. and boundary measure it seems relevant to address the following:

Conjecture [Kapovitch-Ketterer-Sturm '20]

The doubling along the boundary of an $\mathrm{RCD}(K, n)$ space $(X, \mathrm{d}, \mathcal{H}^n)$ is $\mathrm{RCD}(K, n)$.

A key step for the above seems to be the following:

Conjecture [Bruè-Naber-S. '20]

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ with boundary ∂X. Then

$$\Delta \mathrm{d}_{\partial X} \leq -K \mathrm{d}_{\partial X}\,, \quad \text{on } X \setminus \partial X\,.$$

Thank you for your attention!