Ricci Curvature, Fundamental Group, and the Milnor Conjecture (II)

Daniele Semola
FIM-ETH Zürich
daniele.semola@math.ethz.ch

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The Milnor Conjecture

In 1968, Milnor raised the following:

Conjecture

Let (M^n, g) be a smooth, complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Then $\pi_1(M)$ is finitely generated.

In recent joint work with Elia Bruè and Aaron Naber we constructed a family of counterexamples to Milnor's conjecture:

Theorem (Bruè-Naber-S. '23)

For any group $\Gamma < \mathbb{Q}/\mathbb{Z}$ there exists a smooth, complete Riemannian manifold (M^7, g) with $\mathrm{Ric} \geq 0$ and $\pi_1(M) \cong \Gamma$.

State of the art and first open question

- The Milnor conjecture is true in dimension $n = 2$ ([Cohn-Vossen '35]), and $n = 3$ ([Liu '13], see also [Schoen-Yau '82] for the case $\mathrm{Ric} > 0$ and [Pan '18] for a different argument).
- The 7-dimensional counterexamples from [BNS23] trivially extend to any $n \geq 8$.

The following question remains open:

Open question

Does the Milnor conjecture hold in dimensions 4, 5 and 6?

Outline

Setting up the construction

We construct the universal cover $(\tilde{M}, \tilde{g}, \tilde{p})$ together with a free action of Γ by isometries.
The construction is inductive:

- Fix $\Gamma < \mathbb{Q}/\mathbb{Z}$ and a sequence $r_i \to \infty$ with $r_{i+1}/r_i \to \infty$.
- Write $\Gamma = \cup_i \Gamma_i$, with $\Gamma_i < \Gamma_{i+1}$ and all the Γ_i finite.
- In particular, $\Gamma_i =< \gamma_i >$ and $\exists\, k_i \in \mathbb{Z}$ such that $\gamma_i^{k_i} = \gamma_{i-1}$.

Example

Take $\gamma_i = 2^{-i}$ with $k_i = 2$ for every $i \in \mathbb{N}$ to get the dyadic rationals.

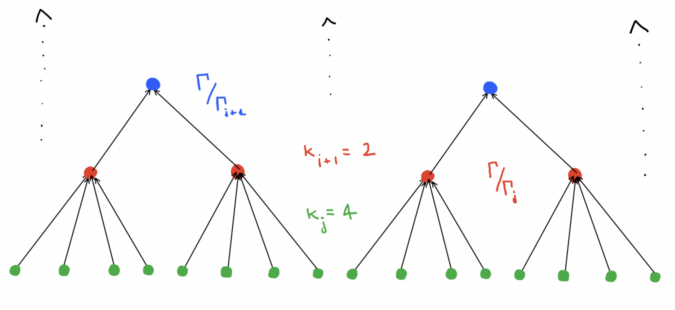
Remark

The Γ_i's are local fundamental groups of M:

$$\Gamma_i =< \gamma \in \Gamma \ : d(\gamma(\tilde{p}), \tilde{p}) \leq r_i \ >< \Gamma \, .$$

A global picture

- Consider
$$\Gamma \times [0, \infty)/_{\sim} \,,$$
where $(\gamma, t) \sim (\gamma', t')$ if $\gamma^{-1}\gamma' \in \Gamma_i$ and $t = t' \geq r_i$ for some $i \in \mathbb{N}$.
- The action of Γ on $\Gamma \times [0, \infty)$ by multiplication on the first factor induces an action of Γ on $\Gamma \times [0, \infty)/_{\sim}$.



From the tree to a manifold

For a global picture:

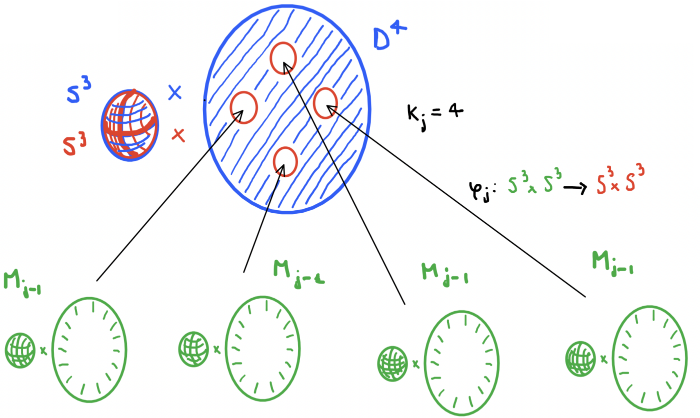
- To obtain $\tilde{M}$, we replace each vertex of the tree with a copy of $S^3 \times D^4$.
- Each edge corresponds to a gluing along boundaries.
- A copy of $S^3 \times D^4$ is glued into another copy of $S^3 \times D^4$ by removing a smaller $S^3 \times D^4$ and gluing the $S^3 \times S^3$ boundaries with a diffeomorphism $\varphi : S^3 \times S^3 \to S^3 \times S^3$.

In the inductive steps we go from $(M_j, g_j, \tilde{p}, \Gamma_j)$ to $(M_{j+1}, g_{j+1}, \tilde{p}, \Gamma_{j+1})$. Roughly speaking,

$$(\tilde{M}, \tilde{g}, \tilde{p}, \Gamma) = \lim_{j \to \infty} (M_j, g_j, \tilde{p}, \Gamma_j) .$$

The inductive step

For the inductive construction: the ends of k_j copies of M_{j-1} are glued into a copy of $S^3 \times D^4$ after removing k_j small copies of $S^3 \times D^4$.



Describing the action

The action of γ_j on the new copies of $S^3 \times D^4$ is:

- by Hopf rotation with angle $2\pi/(k_1 \cdots k_j)$ on S^3;
- by Hopf rotation with angle $2\pi/k_j$ on the D^4-factor.

In particular, it is a sub-action of the $(1, k_1 \cdots k_{j-1})$-Hopf action.

Therefore:

- The action of $\gamma_j^{k_j} (= \gamma_{j-1})$ is by pure rotation on the S^3 factor.

However

- it is induced by the $(1, k_1 \cdots k_{j-2})$-Hopf action on the ends of M_{j-1} that we glue in, by the inductive hypothesis.

Consequence

We need gluing diffeomorphisms φ_j conjugating the two actions:

$$\varphi_j(\theta_{(1,k_1 \ldots k_{j-2})} \cdot (s_1, s_2)) = \theta_{(1,0)} \cdot \varphi_j(s_1, s_2), \quad s_1, s_2 \in S^3 .$$

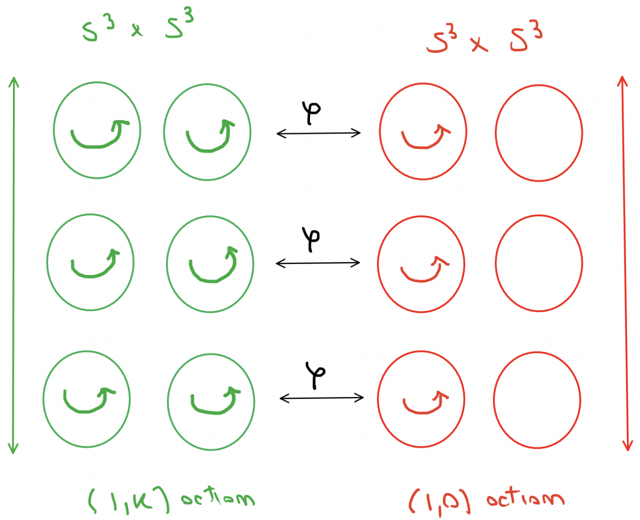
Recap and main challenge

The end of M_{j-1} is diffeomorphic to an annulus in $S^3 \times \mathbb{R}^4 = S^3 \times C(S^3)$, with Γ_{j-1} acting by mixed rotation on both S^3 factors.

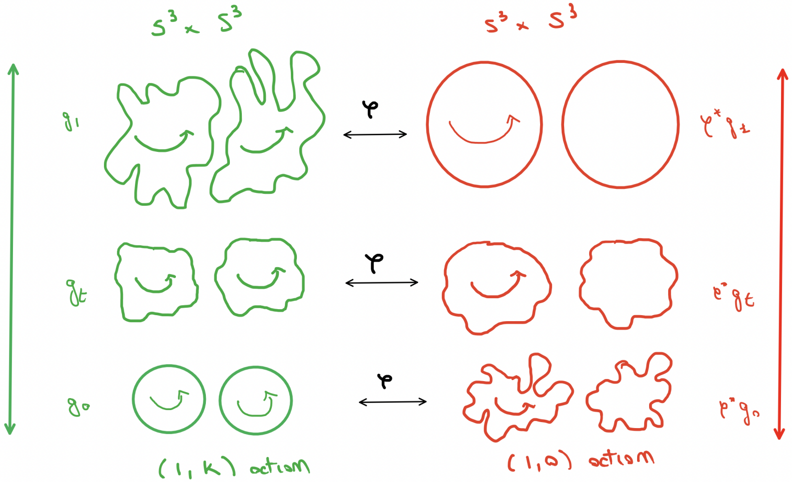
Each of the "lower ends" of the new copy of $S^3 \times D^4 \setminus (\bigcup S^3 \times D^4)$ is diffeomorphic to an annulus in $S^3 \times \mathbb{R}^4 = S^3 \times C(S^3)$.
However, Γ_{j-1} should act by pure rotation on the S^3 factor there.

Main Challenge: we need to twist the ends of M_{j-1} to turn a mixed rotation into a pure rotation on the S^3 factor in a "$\mathrm{Ric} \geq 0$ compatible" way.

The gluing neck, I



The gluing neck, II



Action twisting and positive Ricci curvature

Theorem (Bruè-Naber-S. '23)

Let g_0 be the standard metric on $S^3 \times S^3$ and let $k \in \mathbb{Z}$ be fixed. There exist

a) *a diffeomorphism $\varphi : S^3 \times S^3 \to S^3 \times S^3$;*

b) *a smooth family of Riemannian metrics $(g_t)_{t\in[0,1]}$ on $S^3 \times S^3$;*

such that:

i) $\mathrm{Ric}_t > 0$ *for any $t \in [0,1]$;*

ii) *the S^1-action $\cdot_{(1,k)}$ is isometric on $(S^3 \times S^3, g_t)$ for any $t \in [0,1]$;*

iii) $g_1 = \varphi^* g_0$ *and $\varphi(\theta_{(1,k)}(s_1, s_2)) = \theta_{(1,0)}\varphi(s_1, s_2)$.*

Remark

It is instructive to do an analogous construction for a family of flat metrics on $S^1 \times S^1$.

Comments on the gluing diffeomorphisms

For $k = 1$, we can take (up to isotopy)

$$\varphi(s_1, s_2) = (s_1, s_1^{-1} s_2), \quad s_1, s_2 \in S^3.$$

For general $k \in \mathbb{Z}$, (up to isotopy) φ has the special structure

$$\varphi(s_1, s_2) = (s_1, \psi_{s_1}(s_2)), \quad \psi_{s_1} \in \mathrm{SO}(4).$$

Remark

These gluing diffeomorphisms are not isotopic to the identity.

Remark

Any such φ extends (radially) to a diffeo $\bar{\varphi} : S^3 \times D^4 \to S^3 \times D^4$.

Positive Ricci curvature and $\pi_0(\mathrm{Diff}(S^3 \times S^3))$

Theorem (Bruè-Naber-S. '23)

Let g_0 be the standard metric on $S^3 \times S^3$ and $\varphi \in \mathrm{Diff}(S^3 \times S^3)$. There exists a smooth family of Riemannian metrics g_t on $S^3 \times S^3$ such that:

- $\mathrm{Ric}_t > 0$ *for any* $t \in [0, 1]$;
- $g_1 = \varphi^* g_0$.

Remark

If φ is isotopic to id, the construction is elementary: $g_t := \varphi_t^* g_0$.

Remark

The diffeomorphisms in the previous slide generate $\pi_0(\mathrm{Diff}(S^3 \times S^3))$.

Comments on the bundle structure

- View the S^1-action $(1, k)$ as a multiplication on the left.
- There is a commuting right action of S^3 on $S^3 \times S^3$, by multiplication on the right on the second S^3 factor.
- The induced S^3-action on the quotient $N := S^1 \backslash (S^3 \times S^3)$ is free.
- Moreover $\left(S^1 \backslash (S^3 \times S^3)\right) / S^3 = S^2$.

Lemma

$\pi : N \to S^2$ is a trivial principal S^3-bundle.

Remark

The Riemannian metrics g_0 and g_1 respect this (iterated) bundle structure.

Comments on the geometry, I

Denote by $h_0 = h_0(k)$ and $h_1(k)$ the induced Riemannian metrics on the quotients of $(S^3 \times S^3, g_i)$ w.r.t. the $(1, k)$-action, for $i = 0, 1$.

Remark

$(N^5, h_1(k))$ is isometric to $(S^2 \times S^3, g^{S^2}_{1/2} + g^{S^3}_1)$ for every k.

Remark

As $k \to \infty$, $(N^5, h_0(k)) \to (S^2 \times S^2, g^{S^2}_{1/2} + g^{S^2}_{1/2})$ in the GH sense.

This is an interesting example of collapse with bounded sectional curvature and uniformly positive Ricci curvature (cf. with [Wang-Ziller '90], [Gromov '92] and [Petrunin-Tuschmann '99]).

Comments on the geometry, II

The relevant geometric quantities for the S^1-bundle over the S^3-bundle over S^2 for $t = 0$ and $t = 1$ behave as follows:

	$t = 0$	$t = 1$
S^1-fibers' length	Constant	Constant
Connection of S^1-bundle	Coulomb Gauge	Coulomb Gauge
Curvature of S^1-bundle	Harmonic	Harmonic
Second fund. form of S^3	0	0
Metric on S^3 fibers	Berger sphere	Round
Connection of S^3-bundle	Non-flat	Flat
Metric on base S^2	Round	Round

Deforming the metric on N^5

We interpolate between (N, h_0) and (N, h_1) with a family h_t:

	$t = 0$	$0 < t < 1$	$t = 1$
Second fund. form of S^3	0	0	0
Metric on S^3-fibers	Berger	"Round-off"	Round
Connection of S^3-bundle	Non-flat	"Interpolate"	Flat
Metric on base S^2	Round	Round	Round

Remark

As soon as the metric on the S^3 fiber is shrunk enough, (N, h_t) has $\mathrm{Ric} > 0$, cf. with [Poor '75], [Milnor '76], [Nash '79].

Deforming the metric on $S^3 \times S^3$

Then we deal with the S^1-bundle and construct the family g_t:

	$t = 0$	$0 < t < 1$	$t = 1$
S^1-fibers' length f	Constant	Variable	Constant
Connection of S^1-bundle	Coulomb	Coulomb	Coulomb
Curvature of S^1-bundle ω	Harmonic	Harmonic	Harmonic

Remark

Above the computations are w.r.t the metric h_t on the base N.

For a unit direction U tangent to the S^1-fiber:

$$\mathrm{Ric}(U, U) = -\frac{\Delta f}{f} + \frac{f^2}{2}|\omega|^2 .$$

We can choose the warping function f appropriately to get $\mathrm{Ric} > 0$ although ω might vanish somewhere ([Gilkey-Park-Tuschmann '98]).

Building the neck, I

To build (part of) the gluing neck with $\mathrm{Ric} \geq 0$ from the family g_t with $\mathrm{Ric} > 0$:

- reduce to the case where the metrics g_t have constant volume forms, with an idea going back to [Moser '65].
- Then use an ansatz of the form

$$\mathrm{d}t^2 + f(t)^2 g_t \,,$$

  for a suitably chosen warping function f (cf. [Colding-Naber '11]).

Remark

The reduction to constant volume form cancels out some potentially bad terms in the formulas for the Ricci curvature.

Building the neck, II

Notice that:

- This part of the neck is isometric to $C(S^3 \times S^3)$ on both ends.
- We need to "match" the geometry of $S^3 \times \mathbb{R}^4$ on the gluing regions.

Remark

Need other deformation steps to adjust the sizes of the S^3 factors.

In these regions the metric has a doubly warped product structure:

$$g = dr^2 + f(r)^2 g_{S^3} + h(r)^2 g_{S^3} \; .$$

Asymptotic Geometry and Fund. Groups

Definition (Asymptotic cone/blow-down)

Any pointed Gromov-Hausdorff limit (X, d, q) of a sequence $(M, s_i^{-1}\mathrm{d}_g, p)$ for $s_i \to \infty$ is called asymptotic cone of (M, g).

In general, the asymptotic cones of (M^n, g) with $\mathrm{Ric} \geq 0$:

- don't need to be unique ([Perelman '97]);
- don't need to be metric cones ([Cheeger-Colding '97]).
- don't need to be polar ([Menguy '00]).

Asymptotic Geometry and Fund. Groups

"Regularity" and/or uniqueness of asymptotic cones can force finite generation of the fundamental group.

Theorem (Sormani '00)

If all the asymptotic cones of (M^n, g) with $\mathrm{Ric} \geq 0$ are polar (with respect to the base point) then $\pi_1(M)$ is finitely generated.

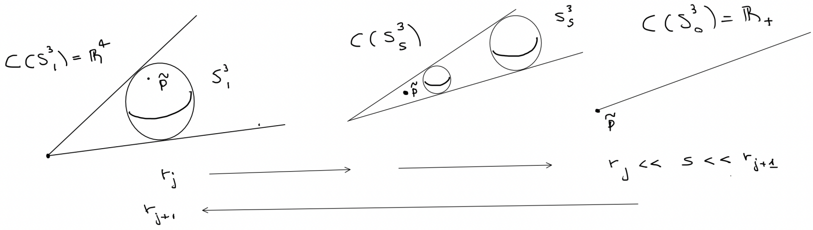
Theorem (Pan '18)

If the asymptotic cone of the universal cover $(\tilde{M}, \tilde{g})$ is unique and a metric cone, then $\pi_1(M)$ is finitely generated.

Asymptotic geom. of the counterexamples, I

We consider the asymptotic geometry of the universal covers $(\tilde{M}, \tilde{g}, \tilde{p})$. "Most" annuli are "mostly" diffeomorphic to $S^3 \times \mathbb{R}^4$:

- The radius of the cross S^3 factor scale invariantly shrinks to 0 as we move towards infinity.
- The radius of the S^3 cross-section for $\mathbb{R}^4 \sim C(S^3)$ scale invariantly oscillates between 0, at the intermediate scales of the gluing necks, and 1, at the scales r_j.



Asymptotic geom. of the counterexamples, II

Lemma

The family of asymptotic cones of (M, g) can include cones over lens spaces $L(k; 1)$ with shrinking radii $s \in [0, 1]$.

The different k's appearing depend on the decomposition $\Gamma = \cup \Gamma_i$.

Remark

A key subtlety is that the cone point is not always the base point, in consistency with [Sormani '00].

Open questions

Question

Does Milnor's conjecture hold for Ricci flat manifolds? (Open in 4d)
Does it hold for Kähler manifolds with $\mathrm{Ric} \geq 0$?

Question

Does Milnor's conjecture hold if the universal cover has Euclidean volume growth?

Question

Can one characterize fundamental groups of open manifolds with $\mathrm{Ric} \geq 0$? (Cf. with [Wei '88] and [Wilking '00]).

Thank you for your attention!