Ricci Curvature, Fundamental Groups, and the Milnor Conjecture

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The Milnor Conjecture

In 1968, John Milnor raised the following:

Conjecture

Let (M^n, g) be a smooth, complete Riemannian manifold with $\mathrm{Ric} \geq 0$. Then $\pi_1(M)$ is finitely generated.

In joint work with Elia Bruè and Aaron Naber we constructed families of counterexamples to Milnor's conjecture:

Theorem (Bruè-Naber-S., March '23)

For any group $\Gamma < \mathbb{Q}/\mathbb{Z}$ there exists a smooth, complete Riemannian manifold (M^7, g) with $\mathrm{Ric} \geq 0$ and $\pi_1(M) \cong \Gamma$.

Theorem (Bruè-Naber-S., November '23)

For any group $\Gamma < \mathbb{Q}/\mathbb{Z}$ there exists a smooth, complete Riemannian manifold (M^6, g) with $\mathrm{Ric} \geq 0$ and $\pi_1(M) \cong \Gamma$.

State of the art and open questions

- The conjecture is true in dimension 2 ([Cohn-Vossen '35]).
- It is true in dimension 3 ([Liu '13], see also [Schoen-Yau '82] for the case $\mathrm{Ric} > 0$ and [Pan '18] for a different argument).
- The counterexamples extend to any dimension ≥ 8.

Open question

Does the Milnor conjecture hold in dimensions 4 and 5?

The Ricci flat case

Open question

Does the Milnor conjecture hold for Ricci flat manifolds?

Remark

The question is open even in dimension 4.

Remark

No single example of a Ricci flat, non-flat 4-manifold with infinite fundamental group is currently known.

Theorem (Anderson-Kronheimer-LeBrun '89)

There exist complete Ricci flat (M^4, g) *with infinitely generated* H_2*.*

Outline

Positive Ricci and fundamental group

Theorem (Bonnet-Myers '41)

If (M^n, g) has $\mathrm{Ric} \geq n-1$, then $\mathrm{diam}(M) \leq \pi$.

By applying the Bonnet-Myers estimate to the universal cover $(\tilde{M}, \tilde{g})$ we immediately get:

Corollary

If (M^n, g) has $\mathrm{Ric} \geq n-1$, then $\pi_1(M)$ is finite.

Bishop-Gromov and polynomial growth

Recall that on (M^n, g) with $\mathrm{Ric} \geq 0$, the function

$$r \mapsto \frac{\mathrm{vol}(B_r(p))}{\omega_n r^n}$$

is non-increasing, by Bishop-Gromov's theorem.

Theorem (Milnor '68)

Let (M^n, g) be complete with $\mathrm{Ric} \geq 0$. Then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of order $\leq n$.

Remark

This rules out the free group $\mathbb{F}_2$, and $\mathbb{Z}^{n+1}$ as possible fundamental groups of (M^n, g) with $\mathrm{Ric} \geq 0$.

Structure of fin. gen. subgroups of $\pi_1(M)$

By [Gromov '81] fin. gen. subgroups of $\pi_1(M)$ are virtually nilpotent.

After [Fukaya-Yamaguchi '92], [Kapovitch-Petrunin-Tuschmann '10]:

Theorem (Kapovitch-Wilking '11)

There exists $C(n) > 0$ s.t. for any (M^n, g) with $\mathrm{Ric} \geq 0$, $\pi_1(M)$ has a nilpotent subgroup N of index $\leq C(n)$ such that any finitely generated subgroup of N

- *is generated by $C(n)$ elements;*
- *has nilpotency length $\leq n$.*

Corollary

$(\mathbb{Z}/k\mathbb{Z})^N$ *is not an admissible π_1 of (M^n, g) with $\mathrm{Ric} \geq 0$ for $N >> n$.*

Existence results

Building on the earlier [Wei '88]:

Theorem (Wilking '00)

For any finitely generated, virtually nilpotent group Γ *there exists a smooth, complete* (M, g) *with* $\mathrm{Ric} \geq 0$ *such that* $\pi_1(M) \cong \Gamma$*.*

Open question

Is the Heisenberg group with rational coefficients $(H_3(\mathbb{Q}), \cdot)$,

$$H_3(\mathbb{Q}) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Q} \right\}.$$

the fundamental group of some complete (M, g) with $\mathrm{Ric} \geq 0$?

Wilking's reduction

It is possible to reduce Milnor's conjecture to the case of abelian fundamental groups:

Theorem (Wilking '00)

Let (M^n, g) be such that $\mathrm{Ric} \geq 0$. Then $\pi_1(M)$ is finitely generated if and only if any abelian subgroup of $\pi_1(M)$ is finitely generated.

Remark

Any $\Gamma < \mathbb{Q}/\mathbb{Z}$ is abelian and it has cyclic finitely generated subgroups.

Remark

Groups $\Gamma < \mathbb{Q}/\mathbb{Z}$ or $\Gamma < \mathbb{Q}$ are indeed the simplest choices for the fundamental group of a potential counterexample.

A few positive results

Theorem (Gromov '78)

Let (M^n, g) be complete with $\mathrm{Sec} \geq 0$. Then $\pi_1(M)$ is generated by at most 3^n elements.

By Bishop-Gromov, (M^n, g) with $\mathrm{Ric} \geq 0$ has at most Euclidean volume growth.

Theorem (Li '86, Anderson '90)

If (M^n, g) with $\mathrm{Ric} \geq 0$ has Euclidean volume growth, $\pi_1(M)$ is finite.

Calabi and Yau proved that if (M^n, g) with $\mathrm{Ric} \geq 0$ is non-compact, then its volume growth is at least linear.

Theorem (Sormani '00)

If (M^n, g) with $\mathrm{Ric} \geq 0$ has linear volume growth, then $\pi_1(M)$ is finitely generated.

Manifolds with infinitely gen. fund. groups

A classical example (compatible with the known restrictions):

Theorem (Steenrod '43)

There exists M^3 with $\pi_1(M)$ isomorphic to the dyadic rationals.

Steenrod credits Vietoris for the idea; cf. with Whitehead manifold.

Remark

The dyadic solenoid complement was a potential Milnor counterexample before [Liu '13], cf. with [Shen-Sormani '06].

Theorem (Folklore?)

Any countable group is the fundamental group of a 5-manifold.

Setting up the construction

We construct the universal cover $(\tilde{M}, \tilde{g}, \tilde{p})$ together with a prop. discont. action of Γ by isometries.
The construction is inductive:

- Fix a sequence $r_i \to \infty$ with $r_{i+1}/r_i \to \infty$.
- Write $\Gamma = \cup_i \Gamma_i$, with $\Gamma_i < \Gamma_{i+1}$ and all the Γ_i finite.
- In particular, $\Gamma_i =< \gamma_i >$ and $\exists\, k_i \in \mathbb{Z}$ such that $\gamma_i^{k_i} = \gamma_{i-1}$.

Example

Take $\gamma_i = 2^{-i}$ with $k_i = 2$ for every $i \in \mathbb{N}$ to get the dyadic rationals.

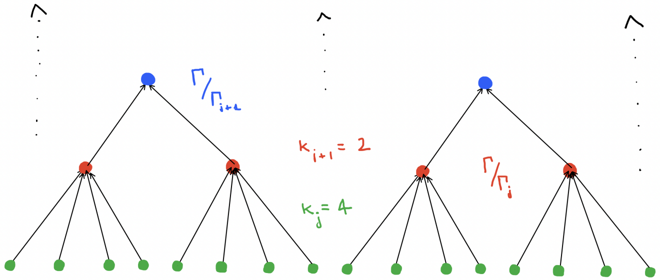
Remark

The Γ_i's are local fundamental groups of M:

$$\Gamma_i =< \gamma \in \Gamma \ : d(\gamma(\tilde{p}), \tilde{p}) \leq r_i \ >< \Gamma\,.$$

The global picture: a tree

- Consider
$$\Gamma \times [0, \infty)/_\sim\,,$$
where $(\gamma, t) \sim (\gamma', t')$ if $\gamma^{-1}\gamma' \in \Gamma_i$ and $t = t' \geq r_i$ for some $i \in \mathbb{N}$.
- The action of Γ on $\Gamma \times [0, \infty)$ by multiplication on the first factor induces an action of Γ on $\Gamma \times [0, \infty)/_\sim$.



From the tree to a manifold

For a global picture:

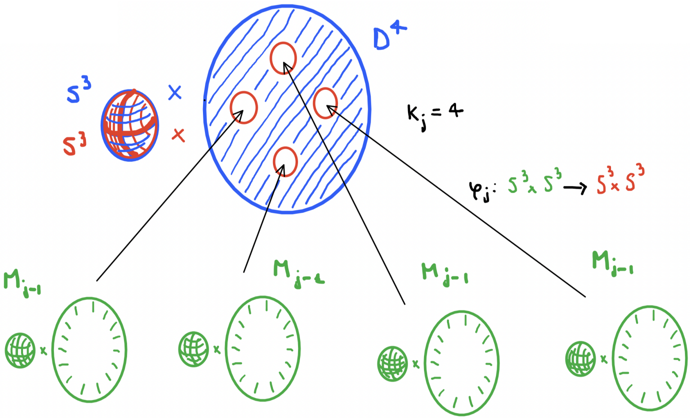
- To obtain $\tilde{M}$, we replace each vertex of the tree with a copy of $S^3 \times D^4$.
- Each edge corresponds to a gluing along boundaries.
- A copy of $S^3 \times D^4$ is glued into another copy of $S^3 \times D^4$ by removing a smaller $S^3 \times D^4$ and gluing the $S^3 \times S^3$ boundaries with a diffeomorphism $\varphi : S^3 \times S^3 \to S^3 \times S^3$.

In the inductive steps we go from $(M_j, g_j, \tilde{p}, \Gamma_j)$ to $(M_{j+1}, g_{j+1}, \tilde{p}, \Gamma_{j+1})$. Roughly speaking,

$$(\tilde{M}, \tilde{g}, \tilde{p}, \Gamma) = \lim_{j \to \infty} (M_j, g_j, \tilde{p}, \Gamma_j) .$$

The inductive step

For the inductive construction: the ends of k_j copies of M_{j-1} are glued into a copy of $S^3 \times D^4$ after removing k_j small copies of $S^3 \times D^4$.



Preliminaries on the action

Remark

There is a free S^1-action on S^3, inducing the Hopf fibration:

$$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2), \quad \theta \in S^1, \ (z_1, z_2) \in S^3 \subset \mathbb{C}^2.$$

Definition

For $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ we denote by (a, b)-Hopf action the induced S^1-action on $S^3 \times S^3$ defined by

$$\theta_{(a,b)} \cdot (s_1, s_2) = (a\theta \cdot s_1, b\theta \cdot s_2), \quad \theta \in S^1, \quad s_1, s_2 \in S^3.$$

Remark

When a, b are coprime the (a, b)-Hopf action is free.

Describing the action

The action of γ_j on the new copies of $S^3 \times D^4$ is:

- by Hopf rotation with angle $2\pi/(k_1 \cdots k_j) = 2\pi/\mathrm{ord}(\gamma_j)$ on S^3;
- by Hopf rotation with angle $2\pi/k_j$ on the D^4-factor.

In particular, it is a sub-action of the $(1, k_1 \cdots k_{j-1})$-Hopf action.

Therefore:

- The action of $\gamma_j^{k_j} (= \gamma_{j-1})$ is by pure rotation on the S^3 factor.

However

- it is induced by the $(1, k_1 \cdots k_{j-2})$-Hopf action on the ends of M_{j-1} that we glue in, by the inductive hypothesis.

Consequence

We need gluing diffeomorphisms φ_j conjugating the two actions:

$$\varphi_j(\theta_{(1,k_1\ldots k_{j-2})} \cdot (s_1, s_2)) = \theta_{(1,0)} \cdot \varphi_j(s_1, s_2), \quad s_1, s_2 \in S^3 .$$

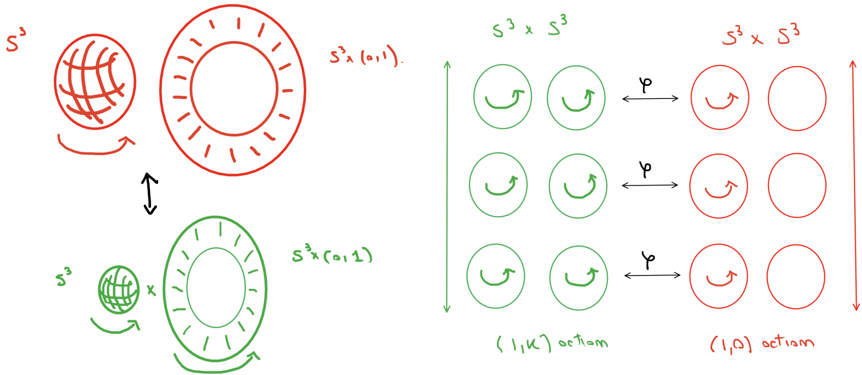
Recap and main challenge

The end of M_{j-1} is diffeomorphic to an annulus in $S^3 \times \mathbb{R}^4 = S^3 \times C(S^3)$, with Γ_{j-1} acting by mixed rotation on both S^3 factors.

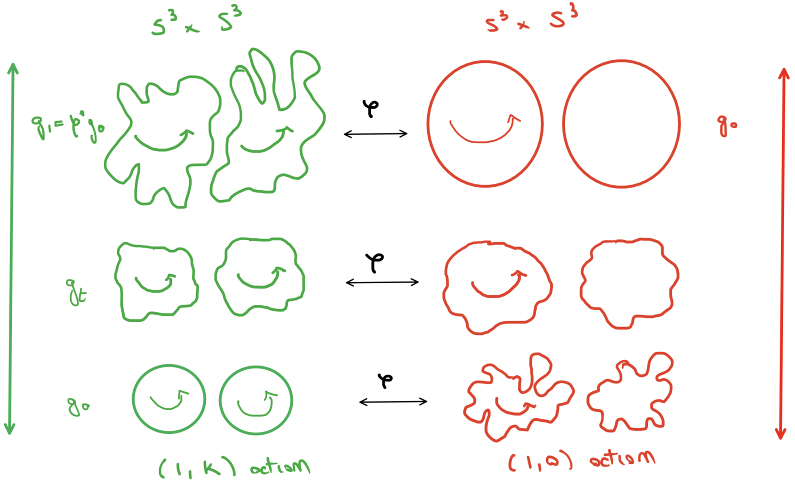
Each of the "lower ends" of the new copy of $S^3 \times D^4 \setminus (\bigcup S^3 \times D^4)$ is diffeomorphic to an annulus in $S^3 \times \mathbb{R}^4 = S^3 \times C(S^3)$.
However, Γ_{j-1} should act by pure rotation on the S^3 factor there.

Main Challenge: we need to twist the ends of M_{j-1} to turn a mixed rotation into a pure rotation on the S^3 factor in a "$\mathrm{Ric} \geq 0$ compatible" way.

The gluing neck, I



The gluing neck, II



Action twisting and positive Ricci curvature

Theorem

Let g_0 be the standard metric on $S^3 \times S^3$ and let $k \in \mathbb{Z}$ be fixed. There exist

a) *a diffeomorphism $\varphi : S^3 \times S^3 \to S^3 \times S^3$;*
b) *a smooth family of Riemannian metrics $(g_t)_{t \in [0,1]}$ on $S^3 \times S^3$;*

such that:

i) $\mathrm{Ric}_t > 0$ *for any $t \in [0,1]$;*
ii) *the S^1-action $\cdot_{(1,k)}$ is isometric on $(S^3 \times S^3, g_t)$ for any $t \in [0,1]$;*
iii) $g_1 = \varphi^* g_0$ *and $\varphi(\theta_{(1,k)}(s_1, s_2)) = \theta_{(1,0)}\varphi(s_1, s_2)$.*

Remark

It is instructive to do an analogous construction for a family of flat metrics on $S^1 \times S^1$.

Comments on the gluing diffeomorphisms

For $k = 1$, we can take (up to isotopy)

$$\varphi_1(s_1, s_2) = (s_1, s_1^{-1} s_2), \quad s_1, s_2 \in S^3.$$

For general $k \in \mathbb{Z}$, (up to isotopy) φ has the special structure

$$\varphi_k(s_1, s_2) = (s_1, \psi_{s_1}(s_2)), \quad \psi_{s_1} \in \mathrm{SO}(4).$$

Remark

These gluing diffeomorphisms are not isotopic to the identity.

Remark

Any such φ extends (radially) to a diffeo $\bar{\varphi} : S^3 \times D^4 \to S^3 \times D^4$.

Theorem

The universal covers of the counterexamples are diffeo. to $S^3 \times \mathbb{R}^4$.

Positive Ricci curvature and $\pi_0(\mathrm{Diff}(S^3 \times S^3))$

Theorem

Let g_0 be the standard metric on $S^3 \times S^3$ and $\varphi \in \mathrm{Diff}(S^3 \times S^3)$. There exists a smooth family of Riemannian metrics g_t on $S^3 \times S^3$ such that:

- $\mathrm{Ric}_t > 0$ *for any* $t \in [0, 1]$;
- $g_1 = \varphi^* g_0$.

Remark

If φ is isotopic to id, the construction is elementary: $g_t := \varphi_t^* g_0$.

Proof.

The diffeomorphisms in the previous slide generate $\pi_0(\mathrm{Diff}(S^3 \times S^3))$, [Kreck '78], [Krylov '03]. □

The 6-dimensional case

The construction of the 6-dimensional counterexamples is analogous, up to replacing $S^3 \times D^4$ with $S^3 \times D^3$ (and hence $S^3 \times S^3$ with $S^3 \times S^2$).

Remark

Constructing the equivariant interpolation of metrics with $\mathrm{Ric} > 0$ on $S^3 \times S^2$ is considerably more delicate than in the $S^3 \times S^3$ case.

Remark

The main reason is that $2 \neq 3$.

The S^1-bundles $\pi'_{(1,k)} : S^3 \times S^2 \to S^1 \backslash \left(S^3 \times S^2\right)$ have:

- fibers with non-constant length;
- non-harmonic curvature 2-form,

contrary to the case of $\pi_{(1,k)} : S^3 \times S^3 \to S^1 \backslash \left(S^3 \times S^3\right)$.

Final remarks

- The asymptotic geometry at infinity of the counterexamples is particularly rich.
  We obtain the first example of (M, g) with $\mathrm{Ric} \geq 0$ with a blow-down which is not simply connected.
- The volume growth of the universal covers is not Euclidean.
  The conjecture is still open in the case of universal covers with Euclidean volume growth.
- The conjecture is open for Kähler manifolds with $\mathrm{Ric} \geq 0$, even in the case of complex surfaces.
- The construction of counterexamples in dimension ≤ 5, if they exist, will most likely require a new method.

Thank you for your attention!