REGULARITY THEORY OF SPACES WITH LOWER RICCI CURVATURE BOUNDS

DANIELE SEMOLA

Contents

1. Lecture 1

1.1. **Introduction and motivations.** We shall consider spaces with lower bounds on their Ricci curvature and upper bounds on their dimension. The main motivating question is:

- how does a complete Riemannian manifold (M^n, g) with $\mathrm{Ric} \geq K$ look like?

Of course there are much more, as:

- what is the meaning of Ricci curvature?

The study of this question is augmented by the introduction of notions of convergence and associated compactness theorems that enable us to compactify certain collections of complete Riemannian manifolds with suitable curvature constraints by adding metric (measure) spaces that are quite singular, in general.

Remark 1.1. Manifolds (spaces) with nonnegative Ricci might be thought as subharmonic functions. Ricci flat manifolds might be thought as harmonic functions. Spaces with nonnegative sectional might be thought as convex functions. Spaces with zero sectional (i.e. flat) correspond to affine functions in this analogy, cf. with [62]. The key issue is that these are highly non linear PDEs.

More in general it is possible to adopt a synthetic perspective, after introducing a notion of metric (measure) space $(X, \mathsf{d}, \mathfrak{m})$ with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $1 \le N < \infty$. These are the so-called $\mathrm{RCD}(K, N)$ metric measure spaces.

Remark 1.2. Cf. with the discussion in [30, Appendix 2]. By *synthetic* we mean that the set of conditions defining the subclass of metric (measure) spaces should not depend on the existence of an underlying smooth structure, nor make any reference to the notion of smoothness. The importance of considering metric *measure* spaces in the setting of lower Ricci bounds was evident from [51].

Both perspectives carry deep insights and we shall adopt a combination of them in these lectures.

We mention the following striking applications of the developments of the theory that we will partially review in these lectures.

The first one is the proof of the Margulis lemma for manifolds with lower Ricci curvature bounds, by Kapovitch-Wilking [70], who settled a conjecture of Gromov.

Definition 1.3. A group G is said to be nilpotent if it admits a central series of finite length. That is a series of normal subgroups

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G\,, \tag{1.1}$$

where $G_{i+1}/G_i \le Z(G/G_i)$ or equivalently $[G, G_{i+1}] \le G_i$.

Definition 1.4. A generator system $b_1, \dots, b_n$ of a group G is called a nilpotency basis if the commutator $[b_i, b_j]$ is contained in the subgroup $< b_1, \dots, b_{i-1} >$ for any $1 \le i < j \le n$.

Theorem 1.5. *In each dimension n there are positive constants $C(n)$, $\varepsilon(n)$ such that the following holds for any complete n-dimensional Riemannian manifold (M, g) with $\mathrm{Ric} \ge -(n-1)$ on a metric ball $B_1(p)$.*

The image of the natural homomorphism

$$\pi_1(B_\varepsilon(p), p) \to \pi_1(B_1(p), p) \tag{1.2}$$

contains a nilpotent subgroup N of index less than $c(n)$. Moreover N has a nilpotent basis of length less than n.

Here is a related conjecture due to Fukaya-Yamaguchi [52].

Conjecture 1.6. The fundamental group of an n-manifold with nonnegative sectional curvature contains a finite index abelian subgroup whose index is bounded by a constant $c(n)$ and whose minimal number of generators is not greater than n.

The second one is the recent proof of the L^2-bounded curvature conjecture for noncollapsed manifolds with bounded Ricci curvature by Jiang-Naber [68].

Theorem 1.7. *In each dimension n for any $v > 0$ there exists a positive constant $C(n, v)$ such that for any complete Riemannian manifold (M, g, p) with*

$$|\mathrm{Ric}| \le (n-1)\,, \quad \mathrm{vol}(B_1(p)) \ge v\,, \tag{1.3}$$

it holds

$$\fint_{B_1(p)} |\mathrm{Riem}|^2 \,\mathrm{dvol} \le C(n, v)\,. \tag{1.4}$$

Remark 1.8. The L^2 bound is sharp, in the sense that no L^p bound for $p > 2$ can be expected to hold. Moreover, the lower volume bound is a crucial assumption, as the L^2 curvature bound fails under a uniform noncollapsing assumption.
The example illustrating the sharpness of the L^2-integrability is the Eguchi-Hanson metric, which is a Ricci flat metric on the cotangent bundle of $\mathbb{S}^2$, $\mathcal{E} = (T^*\mathbb{S}^2, g)$. The rescaled spaces $\mathcal{E}_\eta := (T^*\mathbb{S}^2, \eta g)$ converge in the Gromov-Hausdorff sense as $\eta \to 0$ to $\mathbb{R}^4/\mathbb{Z}_2 = C(\mathbb{RP}^3)$, where $\mathbb{Z}_2$ is acting via $x \mapsto -x$. Using the Chern-Gauss-Bonnet formula it is possible to prove that

$$\int_{\mathcal{E}_\eta} |\mathrm{Riem}|^2 \,\mathrm{dvol} = 16\pi\,, \tag{1.5}$$

independently of η. Moreover, a scaling argument shows that

$$\int_{\mathcal{E}_\eta} |\mathrm{Riem}|^q \,\mathrm{dvol} \to +\infty\,, \quad \text{as } \eta \to 0\,, \tag{1.6}$$

for any $q > 2$.

We address to [68, Example 2.33] for the necessity of the lower volume bound.

A related conjecture due to Yau is the following.

Conjecture 1.9. In each dimension n for any $v > 0$ there exists a positive constant $C(n, v)$ such that for any complete Riemannian manifold (M, g, p) with

$$\mathrm{Ric} \geq -(n-1)\,, \quad \mathrm{vol}(B_1(p)) \geq v\,, \tag{1.7}$$

it holds

$$⨍_{B_1(p)} |\mathrm{Scal}| \,\mathrm{dvol} \leq C(n, v)\,. \tag{1.8}$$

Exercise 1.10. Construct a sequence of $2d$ Riemannian metrics g_n on the plane $\mathbb{R}^2$ with nonnegative Gaussian curvature and such that

$$\int_{B_1(0)} |\mathrm{Scal}|^p \,\mathrm{dvol}_n \to +\infty\,, \tag{1.9}$$

as $n \to \infty$. Idea: the metrics should be rotationally symmetric smoothings of a cone with a singular point. Then Scal_n converges to a Dirac delta at the origin, up to constants and no L^p bound for $p > 1$ is possible.

Remark 1.11. In analogy with the more classical Euclidean theory of elliptic PDEs, a priori estimates can be useful to establish existence of certain solutions. The fundamental difference with the Euclidean case is the absence of a fixed background metric with known properties.

1.2. **Lagrangian *vs* Eulerian.** In practice, Ricci curvature appears (at the very least) from two points of view: estimates on the Jacobian determinant of the exponential map or Bochner's formula. They are complimentary points of view on the same phenomenon [95, Chapter 14]. In the Eulerian point of view we deal with gradients, Laplacians, Hessians, in the Lagrangian point of view we deal with curves in the ambient space. See also [26, Chapter 2].

In the synthetic theory of lower Ricci bounds there will be a Lagrangian approach based on Optimal Transport (the Sturm-Lott-Villani theory) and a Eulerian approach based on Γ-calculus (the Bakry-Émery theory).

In the Lagrangian approach we move along a path $\gamma(t)$ keeping track of the initial position $\gamma(0)$. In the Eulerian approach the focus is on the velocity $\xi(t, x)$. In order to switch from Lagrangian to Eulerian we set

$$\dot\gamma(t) = \xi(t, \gamma(t))\,. \tag{1.10}$$

The duality in general is not globally well defined. This is well known to those familiar with fluid dynamics, which is where the Eulerian/Lagrangian duality comes from.

1.2.1. *Classical computations with Jacobi fields.* We refer to [62, Section 5], [88, Chapter 2, Section4] and [95, Chapter 14] for this introductory part.

We consider equidistant hypersurfaces from a given hypersurface Σ in a Riemannian manifold (M^N, g). The key object is the distance function d_Σ. This is not smooth, however it solves the equation

$$|\nabla \mathsf{d}_\Sigma| = 1 \tag{1.11}$$

in suitable sense. We adopt the shortened notation $r := \mathsf{d}_\Sigma$. Locally it is possible to write $g = \mathrm{d}r^2 + g_r$ as a warped product, where g_r is the induced metric on the level set $U_r := \{\mathsf{d}_\Sigma = r\}$. We are interested on the rate of change of g_r when we move away from Σ. This is governed by the second fundamental form, or equivalently by the Hessian of r,

$$\mathrm{Hess}\, r(X, Y) = g(\mathrm{II}(X), Y)\,, \tag{1.12}$$

The idea is that $\mathrm{II} = \nabla \partial_r$ measures how the induced metric g_r on U_r changes by looking at how the unit normal varies.

Theorem 1.12 (Radial curvature equation/Tube formula)**.**

$$\nabla_{\partial r}\mathrm{II} + (\mathrm{II})^2 = -\mathrm{Riem}_{\partial_r}\,. \tag{1.13}$$

Definition 1.13. A Jacobi field J for a smooth distance function r is a vector field independent of r, namely a solution of

$$L_{\partial_r} J = 0\,, \tag{1.14}$$

where L_{∂_r} denotes the Lie derivative with respect to ∂_r. A parallel vector field for r is a solution of

$$\nabla_{\partial_r} X = 0\,. \tag{1.15}$$

Remark 1.14. Jacobi fields satisfy a second order equation known as the Jacobi equation:

$$\nabla_{\partial_r}\nabla_{\partial_r} J = -\mathrm{Riem}(\partial_r, J)\partial_r\,. \tag{1.16}$$

We can evaluate the first radial curvature equation along Jacobi fields and get

$$\partial_r g(J_1, J_2) = 2\,\mathrm{Hess}\, r(J_1, J_2)\,. \tag{1.17}$$

Analogously, we can evaluate the tube formula along parallel vector fields and get

$$\partial_r\left(\mathrm{Hess}\, r(X,Y)\right) + \mathrm{Hess}^2\, r(X,Y) = -\mathrm{Riem}(X, \partial_r, \partial_r, Y)\,. \tag{1.18}$$

Curvature yields information about the Hessian when evaluating the tube formula on parallel vector fields. Then we obtain information about the induced metric by evaluating the first radial curvature equation along Jacobi fields.

The Ricci curvature appears when we trace the equations above.

Definition 1.15. The mean curvature of U_r is the trace of the second fundamental form, equivalently, the sum of the principal curvatures.

Remark 1.16. As discussed above, the second fundamental form measures the rate of change of the induced metric. The mean curvature measures the rate of change of the induced volume form.

We can consider the normal geodesic map T_r from Σ to U_r pushing any point $x \in \Sigma$ into $\exp_x(r\nabla \mathsf{d}_\Sigma) \in U_r$. Then we look at the induced volume form vol_r on U_r and pullback on Σ via T_r to obtain vol_r^*. The rate of change of the induced volume form is measured by the mean curvature:

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{vol}_r^* = H_r \mathrm{vol}_r^*\,. \tag{1.19}$$

We can equivalently compute the rate of change of the Jacobian determinant of the map T_r to get

$$\frac{\mathrm{d}}{\mathrm{d}r}\log JT_r = H_r\,. \tag{1.20}$$

Then we obtain equivalent forms of the traced tube formula:

$$\frac{\mathrm{d}}{\mathrm{d}r}H_r = -\operatorname{tr}\left(\mathrm{II}_r\right)^2 - \mathrm{Ric}(\partial_r,\partial_r) \tag{1.21}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log JT_r = -\operatorname{tr}\left(\mathrm{II}_r\right)^2 - \mathrm{Ric}(\partial_r,\partial_r)\,. \tag{1.22}$$

If we wish to exploit finite dimensionality, then we can estimate

$$\operatorname{tr}\left(\mathrm{II}_r\right)^2 \geq \frac{\left(\operatorname{tr}(\mathrm{II}_r)\right)^2}{N-1}\,. \tag{1.23}$$

Remark 1.17. Notice that here it would be simpler to estimate

$$\operatorname{tr}\left(\mathrm{II}_r\right)^2 \geq \frac{\left(\operatorname{tr}(\mathrm{II}_r)\right)^2}{N}\,. \tag{1.24}$$

In order for the refined estimate to work we separate the direction of motion. This is very hard to achieve on non smooth spaces, cf. with the discussion below, as it needs a splitting between normal and tangent directions.

Hence

$$\frac{\mathrm{d}}{\mathrm{d}r}H_r \leq -\frac{\left(\operatorname{tr}(\mathrm{II}_r)\right)^2}{N-1} - \mathrm{Ric}(\partial_r,\partial_r)\,, \tag{1.25}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log JT_r \leq -\frac{\left(\operatorname{tr}(\mathrm{II}_r)\right)^2}{N-1} - \mathrm{Ric}(\partial_r,\partial_r)\,. \tag{1.26}$$

Remark 1.18. The mean curvature of the equidistant hypersurface corresponds to the Laplacian of the distance function:

$$H_r = \Delta\mathsf{d}_\Sigma := \operatorname{div}\nabla\mathsf{d}_\Sigma\,. \tag{1.27}$$

This is perfectly fine when d_Σ is smooth and can be given a meaning in general.

Remark 1.19. If we assume that $\mathrm{Ric}\geq 0$ and neglect the dimensional term, then (1.26) turns into a clean concavity inequality for the Jacobian determinant. Hence it admits a *synthetic* equivalent formulation not involving any derivative.
The same is true, although a bit more technical, for any lower Ricci bound and in the dimensional case too.

1.2.2. *Bochner's identity and Bochner's inequality.* Cf. with the presentation in Bakry-Gentil-Ledoux's book [14, Appendix C.6].

On a smooth Riemannian manifold (M,g) the Bochner identity holds for sufficiently smooth functions $u: M\to\mathbb{R}$, namely

$$\Delta\frac{1}{2}\left|\nabla u\right|^2 = \left\|\mathrm{Hess}\,u\right\|^2_{\mathrm{HS}} + \nabla u\cdot\nabla\Delta u + \mathrm{Ric}(\nabla u,\nabla u)\,. \tag{1.28}$$

This is often rewritten as

$$\Gamma_2(u) := \Delta\frac{1}{2}\left|\nabla u\right|^2 - \nabla u\cdot\nabla\Delta u = \left\|\mathrm{Hess}\,u\right\|^2_{\mathrm{HS}} + \mathrm{Ric}(\nabla u,\nabla u)\,. \tag{1.29}$$

Here Γ_2 is the iterated Γ operator, as $\Gamma(u) := \frac{1}{2}\Delta\left(u^2\right) - u\Delta u$. More in general we can define

$$\Gamma(f,g) := \frac{1}{2}\left[\Delta(fg) - f\Delta g - g\Delta f\right]\,. \tag{1.30}$$

The terminology *iterated* Γ_2 *operator* comes from the fact that we are replacing the products in (1.30) with the Γ operator itself, as

$$\Gamma_2(f,g) := \frac{1}{2}\left[\Delta\left(\Gamma(f,g)\right) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f)\right] . \tag{1.31}$$

Letting $\lambda_1,\dots,\lambda_N$ be the eigenvalues of $\mathrm{Hess}\, u$, $\|\mathrm{Hess}\, u\|^2_{\mathrm{HS}} = \sum_i \lambda_i^2$ and $\Delta u = \sum_i \lambda_i$. By Cauchy-Schwarz the Bochner identity implies the *dimensional* Bochner inequality

$$\Gamma_2(u) \ge \frac{(\Delta u)^2}{N} + \mathrm{Ric}(\nabla u, \nabla u) \tag{1.32}$$

and the *a-dimensional* Bochner inequality

$$\Gamma_2(u) \ge \mathrm{Ric}(\nabla u, \nabla u) . \tag{1.33}$$

We are going to consider spaces for which the Bakry-Émery condition $\mathrm{BE}(K,N)$ holds, cf. with the seminal [13].

Definition 1.20. Given $K \in \mathbb{R}$ and $1 \le N < \infty$ we say that the $\mathrm{BE}(K,N)$ condition holds if

$$\Gamma_2(u) \ge \frac{(\Delta u)^2}{N} + K\Gamma(u) , \tag{1.34}$$

for any function u in an algebra of test functions $\mathcal{A}$.

Remark 1.21. Here the focus is on Dirichlet forms and symmetric Markov semigroups, rather than on the original datum (Riemannian manifold). This makes quite evident the distinct role of the distance and of the reference measure.

Remark 1.22. It is not hard to verify that a smooth Riemannian manifold (M^N, g) verifies $\mathrm{BE}(K,N)$ if and only if $\mathrm{Ric} \ge Kg$.

Remark 1.23. We can formally apply the Bochner identity/inequality to a distance function, noticing that $|\nabla \mathsf{d}|^2 \equiv 1$ to infer

$$\nabla \mathsf{d} \cdot \nabla \Delta \mathsf{d} = \|\mathrm{Hess}\, \mathsf{d}\|^2_{\mathrm{HS}} + \mathrm{Ric}(\nabla \mathsf{d}, \nabla \mathsf{d}) . \tag{1.35}$$

The left hand side admits an interpretation as derivative of $\Delta \mathsf{d}$ along the gradient flow lines of d. This interpretation brings us back to the Lagrangian perspective presented above, namely to the traced tube formula (1.22).

1.3. **Non smooth spaces with lower Ricci bounds.** Nowadays there is a well developed theory of $\mathrm{RCD}(K,N)$ metric measure spaces. They are triples $(X,\mathsf{d},\mathfrak{m})$, where (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a σ-finite measure. We shall always assume that $\mathfrak{m}$ is fully supported, unless otherwise stated. They can be equivalently characterized from the Lagrangian and the Eulerian view-point.

A notion of energy can be introduced in great generality.

Definition 1.24. Given $f \in L^2(X,\mathfrak{m})$ we let

$$\mathsf{Ch}(f) := \inf\left\{ \liminf_{n\to\infty} \int_X \mathrm{lip}^2 f_n \,\mathrm{d}\mathfrak{m} \,:\, \|f_n - f\|_2 \to 0 \right\} . \tag{1.36}$$

The first requirement is that the Cheeger energy is a quadratic form, i.e. it satisfies the parallelogram rule

$$2\mathsf{Ch}(f) + 2\mathsf{Ch}(g) = \mathsf{Ch}(f+g) + \mathsf{Ch}(f-g) . \tag{1.37}$$

This is made to rule out Finsler geometries. The condition is known as infinitesimal Hilbertianity and it comes from [5, 54], compare also with the discussion by Gromov in [62, Section 5] and by Cheeger-Colding in [30, Appendix 2].

Definition 1.25. The metric measure space $(X,\mathsf{d},\mathfrak{m})$ is said to be infinitesimally Hilbertian provided Ch is a quadratic form in $L^2(X,\mathfrak{m})$.

Remark 1.26. A Banach space is infinitesimally Hilbertian if and only if it is Hilbert. A Finsler manifold is infinitesimally Hilbertian if and only if it is Riemannian. So the quadraticity of the energy is a regularity assumption, cf. with the discussion in Cheeger-Colding's [30, Appendix 2].

Under these assumptions, we can define $|\nabla u|$ and $\nabla u \cdot \nabla v$ at least $\mathfrak{m}$-a.e. for any $u, v \in L^2(X, \mathfrak{m})$ with $\mathsf{Ch}(u), \mathsf{Ch}(v) < \infty$. Let us introduce the notation $H^{1,2}(X, \mathsf{d}, \mathfrak{m}) := L^2(X, \mathfrak{m}) \cap \{\mathsf{Ch} < \infty\}$.

Then we can introduce a notion of Laplacian by integration by parts, i.e. we exploit the identity

$$\int_X g\Delta f \, \mathrm{d}\mathfrak{m} = -\int_X \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m} . \tag{1.38}$$

Definition 1.27. Let $f \in H^{1,2}$. Then we say that f belongs to the domain of the Laplacian, $f \in D(\Delta)$, if and only if there exists a function $h \in L^2$ (which is unique, a posteriori, hence denoted $\Delta f := h$) such that

$$\int_X g\Delta f \, \mathrm{d}\mathfrak{m} = -\int_X \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m} , \quad \text{for any } g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) . \tag{1.39}$$

Remark 1.28. Ch quadratic implies that Δ is linear, which is not true in general.

Remark 1.29. On a weighted Riemannian manifold this gives rise to the weighted/Witten Laplacian, namely if $(M, g, e^{-f}\mathrm{vol})$

$$\Delta u = \Delta_g u - \nabla f \cdot \nabla u . \tag{1.40}$$

This we can check by computing

$$\int_M v(\Delta_g u - \nabla f \cdot \nabla u) \, \mathrm{d}e^{-f}\mathrm{vol}_g = \int_M \nabla v \cdot \nabla u \, \mathrm{d}e^{-f}\mathrm{vol}_g . \tag{1.41}$$

Remark 1.30. On a smooth Riemannian manifold with boundary, endowed with the volume measure, a smooth function belongs to the domain of the Laplacian in the above sense if and only if it satisfies homogeneous Neumann boundary conditions and its Laplacian is L^2.

The first way to talk about Curvature-Dimension bounds is to empoly a weak version of the Γ_2 criterion, after intergating by parts. We say that a infinitesimally Hilbertian metric measure space verifies the $\mathrm{BE}(K, N)$ condition provided the inequality

$$\frac{1}{2}\int_X \Delta\varphi \, |\nabla u|^2 \, \mathrm{d}\mathfrak{m} \geq \int_X \varphi \left(\frac{(\Delta u)^2}{N} + \nabla u \cdot \nabla \Delta u + K \, |\nabla u|^2 \right) \mathrm{d}\mathfrak{m} \tag{1.42}$$

holds for any function $u \in D(\Delta)$ with $\Delta u \in H^{1,2}$ and for any function $\varphi \in D(\Delta) \cap L^\infty$ with $\varphi \geq 0$ and $\Delta\varphi \in L^\infty$.

A further regularity property is required in order to avoid pathological examples. Namely we require that any function $f \in H^{1,2}$ such that $|\nabla f| \leq 1$ $\mathfrak{m}$-a.e. admits a 1-Lipschitz representative.

Example 1.31. Without the *Sobolev to Lipschitz* property, it is possible to construct examples where the local dimension jumps by gluing two Riemannian manifolds along a *small* set, see the work of Honda [67]. The $\mathrm{RCD}(K, N)$ condition prevents this possibility.

The first approach to be developed was based on Optimal Transport instead. The theory of Curvature-Dimension bounds for metric measure spaces by Sturm [93, 94] and independently Lott-Villani [76] came after a series of contributions shedding light on the connection between lower Ricci curvature bounds and Optimal Transport on smooth Riemannian manifolds, [79, 85, 42, 90].

Remark 1.32. The theory was developed with the distance induced by Optimal Transport with quadratic cost $c(x,y)=\mathsf{d}^2(x,y)$ in the first place. A series of recent contributions has shed light onto the independence of the theory of the power of the distance $\mathsf{d}^p(x,y)$, under very general assumptions, see for instance [2].

Integrating on a bunch of geodesics sufficiently spread out the Lagrangian inequalities (1.26), it is possible to relate Curvature-Dimension bounds to convexity properties of non linear functionals on the space of probability measures endowed with the Wasserstein distance induced from the Optimal Transport problem. The prototype is the logarithmic Boltzmann-Shannon entropy:

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \int \rho \log \rho \, \mathrm{d}\mathfrak{m} \,, \qquad \mu = \rho \mathfrak{m} \,. \tag{1.43}$$

Remark 1.33. We have already discussed the infinitesimal version of these convexity properties.

This opens the way to the possibility of defining Curvature-Dimension bounds for metric measure spaces, with no reference to smoothness.

Key features of this approach are:

- second order differential inequalities are integrated up to concavity properties;
- singularities, if present, always go in the right direction, cf. with the discussion in Gromov's lectures [64]. This is in perfect analogy with what happens for convexity. A semiconvex function admits second derivatives in the sense of distributions. They are measures and the singular part is always nonnegative.
- all the directions of motion/displacement can be tested via suitably chosen optimal transportations.

Remark 1.34. As pointed out before, the Curvature-Dimension condition is not strong enough to rule out normed vector spaces that are not Hilbert. These were known not to appear as Ricci limit spaces after [29].

Remark 1.35. Optimal Transport gives a way to localize with respect to the direction the mean curvature comparison, while keeping it averaged, cf. with the discussion in [30, Appendix 2].

Then $\mathrm{RCD}(K,N)$ spaces are those infinitesimally Hilbertian metric measure spaces for which the Curvature-Dimension condition $\mathrm{CD}(K,N)$ holds, see [5] and [54].

The connections between the two approaches (Lagrangian and Eulerian) have been clarified in a series of recent contributions. Ambrosio-Gigli-Savaré [5, 6] deal with the equivalence for $N=\infty$ (see also [4]). Then Erbar-Kuwada-Sturm [47] and Ambrosio-Mondino-Savaré [9] deal with the finite dimensional approach, proving equivalence between $\mathrm{BE}(K,N)$ and $\mathrm{RCD}^*(K,N)$. Equivalence with $\mathrm{RCD}(K,N)$ stood open until the work of Cavalletti-Milman who managed to single out the direction of motion in the synthetic setting, cf. with Remark 1.17.

Remark 1.36. Compatibility with the smooth case holds, i.e. smooth Riemannian manifolds with lower Ricci bounds are RCD spaces. Compatibility with the synthetic theory of Alexandrov spaces with lower sectional curvature bounds holds true, this is due to Petrunin for nonnegative sectional curvature and later refined by Zhang-Zhu in the case of general lower curvature bounds.

2. Lecture 2

In the next few sections we discuss some mild regularity properties of spaces with lower Ricci curvature bounds.

2.1. **Laplacian comparison, Bishop-Gromov and Poincaré inequalities.** We shall denote by $(M_{N,K}, g_{N,K})$ the (unique) simply connected complete Riemannian manifold with constant sectional curvature equal to K and dimension N. Notice that the Ricci curvature of $M_{N,K}$ verifies $\mathrm{Ric}_{M_{N,K}} = (N-1)Kg$. For $K > 0$, $M_{N,K}$ is a sphere, for $K = 0$ it is the Euclidean space, for $K < 0$ it is the hyperbolic space.

Recall that the Riemannian metrics on these models can be written as warped product metrics

$$g_{N,K} = \mathrm{d}r^2 + f_{K,N}^2 \tilde{g}\,, \tag{2.1}$$

where $\tilde{g}$ is the canonical metric on the sphere $\mathbb{S}^{N-1}$ with constant sectional curvature 1.

Remark 2.1. A meaningful mean curvature comparison should hold with no restriction to the regular locus of the distance function, as soon as mean convexity is suitably interpreted, cf. with Gromov's lecture [62, Section 5]. The key insight is that we want the intersection of two mean convex domains to be mean convex, even though smoothness gets lost in general. Moreover, there is the possibility to test mean convexity with smooth touching mean convex hypersurfaces.

2.1.1. *Laplacian comparison.* The Lagrangian computations discussed above, easily imply Laplacian comparison theorems for the distance function from a point, away from the cut-locus. If $\mathrm{Ric} \geq 0$ on a smooth N-dimensional Riemannian manifold (M, g) and $p \in M$, then

$$\Delta \mathsf{d}_p \leq \frac{N-1}{\mathsf{d}_p}\,, \tag{2.2}$$

away from the cut-locus. This uses the Lagrangian computation along minimizing geodesics, together with the expression for the intial asymptotic near p, that can be easily obtained as Riemannian manifolds are locally Euclidean. Notice that by chain rule, the above is equivalent to

$$\Delta \mathsf{d}_p^2 \leq 2N\,. \tag{2.3}$$

Moreover, locally on a smooth manifold $\Delta \mathsf{d}_p^2 = 2N + O(\mathsf{d}_p^2)$ in a small neighbourhood of a point. Starting from the work of Calabi, see in particular [22, Theorem 3], it was evident that global Laplacian comparison theorems, valid up to the cut-locus, were necessary.

Remark 2.2. The main motivation in [22] was to obtain maximum principles for functions that are not differentiable but satisfy weakly a second order partial differential inequality. The distance function is a prototype.

Remark 2.3. The function $x \mapsto \sqrt{|x|}$ is concave on $(-\infty, 0)$ and on $(0, \infty)$ (in particular it is concave a.e.) but it admits an interior minimum point. The minimum principle fails and we need a stronger notion of superharmonicity to get useful conclusions.

The perspective adopted in [22] was the one of barriers. If we can put above our function (possibly not differentiable) smooth functions with Laplacian slightly bigger than the term in the Laplacian comparison, then the Laplacian comparison holds in the sense of barriers.

Definition 2.4. A smooth function f is an upper barrier for u at x if $f \geq u$ and $f(x) = u(x)$. The Laplacian comparison $\Delta u \leq c$ holds in the sense of barriers provided for any $\eta > 0$ and any x we can find an upper barrier f_η at x with $\Delta f_\eta \leq c + \eta$.

Later in the proof of the splitting theorem for manifolds with nonnegative Ricci curvature by Cheeger-Gromoll [33] the perspective was that of comparison with harmonic functions with the same boundary data, on any domain. See also Cheeger's lectures [26, Theorem 4.1].

Theorem 2.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(0, N)$ *metric measure space. Let $p \in X$. Then the Laplacian comparison*

$$\Delta \mathsf{d}_p \leq \frac{N-1}{\mathsf{d}_p} \tag{2.4}$$

holds away from p in the weak sense. This amounts to say that

$$-\int_X \nabla f \cdot \nabla \mathsf{d}_p \, \mathrm{d}\mathfrak{m} \leq \int_X (N-1) \frac{f}{\mathsf{d}_p} \, \mathrm{d}\mathfrak{m} \,, \tag{2.5}$$

for any function $f \in H^{1,2}$ supported in $X \setminus \{p\}$.

Remark 2.6. On $\mathbb{R}^N$, $\Delta \mathsf{d}_p = (N-1)/\mathsf{d}_p$. Analogous statements hold for general $K \in \mathbb{R}$, with comparison with the Laplacian of the distance from points in the model spaces.

Remark 2.7. For proving the Laplacian comparison in the sense of distributions on smooth manifolds the idea is to show that the singular contribution coming from the cut-locus is a nonpositive measure by approximation, working outside a tubular neighbourhood of the cut-locus and sending the size to 0.

Let us point out the work of Greene-Wu [61] for various equivalence results between notions of subharmonic functions for Riemannian manifolds. In particular, the distributional approach is equivalent to the approach via comparison with solutions of the Dirichlet problem.

2.1.2. *Bishop-Gromov volume monotonicity.* The following volume comparison was proved for smooth Riemannian manifolds with lower Ricci bounds by Bishop, for radii smaller than the injectivity radius. Gromov [63, 5.3bis Lemma] extends it to a global comparison result.

Theorem 2.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *metric measure space. Let $p \in X$. Then the following relative volume comparison holds. The function*

$$r \mapsto \frac{\mathfrak{m}(B_r(p))}{\mathrm{vol}(B_r(\bar{p}))} \tag{2.6}$$

is monotone decreasing, where $B_r(\bar{p})$ is the ball in the model space with dimension N and constant sectional curvature $K/(N-1)$.

We will often denote by $v_{N,K}(r)$ the volume of the ball of radius $r > 0$ in the simply connected model space with dimension N and constant sectional curvature $K/(N-1)$.

Remark 2.9. There are proofs that exploit the Curvature-Dimension condition formulated in terms of Optimal Transport, see for instance [95, Theorem 30.11]. The idea is to apply the concavity inequality for entropies along optimal transport between the Dirac delta at the centre of the ball and approximations of the surface measure of a sphere.

On a smooth Riemannian manifold, the Bishop-Gromov volume monotonicity is actually a consequence of the stronger mean curvature comparison that holds along minimizing geodesics emanating from a point. It is classical that the distance function is smooth until we reach the cut locus.

Then we write the volume element in polar coordinates as $\mathrm{dvol} = \mathcal{A}(r, \theta) \, \mathrm{d}r$ where $\mathcal{A}(r, \theta)$ is a certain volume element on the sphere. The mean curvature comparison amounts to

$$m(r, \theta) := \frac{\mathcal{A}'(r, \theta)}{\mathcal{A}(r, \theta)} \leq \overline{m}(r, \theta) \,, \tag{2.7}$$

where $\overline{m}(r, \theta) = \overline{m}(r)$ is the mean curvature on the model space which admits the explicit expression

$$\overline{m}(r) = (N-1) \frac{f'_{K,N}(r)}{f_{K,N}(r)} \,. \tag{2.8}$$

The mean curvature comparison can be integrated up to the Bishop-Gromov monotonicity by taking into account the initial conditions.

Notice that the expression for the Laplacian in polar coordinates involves the mean curvature, namely

$$\Delta = \frac{\partial^2}{\partial r^2} + m(r)\frac{\partial}{\partial r} + \tilde{\Delta}\,, \tag{2.9}$$

where $\tilde{\Delta}$ is the Laplacian with respect to the induced metric on the r-sphere. This brings us back to the Laplacian comparison.

We present a formal proof of the Bishop-Gromov inequality starting from the Laplacian comparison, when $K = 0$. Generalizations of the outline to any lower Ricci bound are possible. By the coarea formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathfrak{m}(B_r(x))}{\omega_N r^N} = \frac{1}{\omega_N r^N}\left(\mathrm{Per}(B_r(x)) - \frac{N\mathfrak{m}(B_r(x))}{r}\right). \tag{2.10}$$

In order to prove Bishop Gromov we are left to prove that

$$r\,\mathrm{Per}(B_r(x)) \le N\mathfrak{m}(B_r(x))\,. \tag{2.11}$$

To this aim we apply Gauss-Green with vector field $\frac{1}{2}\nabla \mathsf{d}_x^2$. By Laplacian comparison $\mathrm{div}\,\frac{1}{2}\nabla\mathsf{d}_x^2 \le N$. Moreover, $\frac{1}{2}\nabla\mathsf{d}_p^2$ is equal to $r\nu_{B_r(x)}$, where we denoted by $\nu_{B_r(x)}$ the exterior unit normal to the boundary of $B_r(x)$. Hence

$$N\mathfrak{m}(B_r(x)) = \int_{B_r(x)} N\,\mathrm{d}\mathfrak{m} \ge \int_{B_r(x)} \mathrm{div}\,\frac{1}{2}\nabla\mathsf{d}_x^2 = r\,\mathrm{Per}(B_r(x))\,. \tag{2.12}$$

2.1.3. *Poincaré inequality.* A mild regularity property of spaces with lower Ricci bounds is the Poincaré inequality.

The relevance of doubling and Poincaré assumptions for the developments of analysis on metric measure spaces has been clarified in the last thirty years. Very roughly speaking, doubling inequalities control the measure of larger balls with the measure of smaller balls. Poincaré inequalities control the deviation of a function from its average in a smaller ball with its energy on a larger ball.

Remark 2.10. The idea is that knowing a certain estimate on a ball $B_r(x)$ we deduce a better estimate on the ball $B_{r/2}(x)$. The origins can be traced in De Giorgi's solution of the XIX Hilbert problem about Hölder regularity for solutions of elliptic second order differential equations in divergence form. Similar ideas play a role in the independent work of Nash and in the subsequent work of Moser.

Theorem 2.11. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K,N)$ *metric measure space. Then for any $R > 0$ there exists a constant $C(K,N,R) > 0$ such that the local Poincaré inequality*

$$⨍_{B_r(x)}\left|u - ⨍_{B_r(x)} u\right| \mathrm{d}\mathfrak{m} \le C(K,N,R) ⨍_{B_{2r}(x)} |\nabla u|\,\mathrm{d}\mathfrak{m} \tag{2.13}$$

holds for any Lipschitz function $u : X \to \mathbb{R}$, any $x \in X$ and any $0 < r < R$. If $K = 0$, then $C(0,N,R) = C(N)R$ works.

Remark 2.12. Poincaré inequalities are related to the richness of the set of geodesics, cf. with the discussion in Villani's book [95, Chapter 19]. The fundamental idea is that we should be able to transfer mass without making the density blow-up too quickly.

The Curvature-Dimension condition basically provides this property. By restriction-localization it is possible to get uniform density bounds

$$\rho_t \le \frac{C}{t^N} \tag{2.14}$$

for the density of the interpolation between $\mu_0 = \delta_x$ and μ_1 the normalized restriction of $\mathfrak{m}$ to $B_r(x)$. Analogously we can uniformly bound the density of the interpolant given bounds for the density of the endpoints of a Wasserstein geodesic.

Notice that in the smooth setting, these bounds would follow from the Jacobian estimates described near to (1.26), via elementary one dimensional considerations.

The key idea to get Poincaré from the density bounds for the interpolation (2.14) is to estimate

$$|u(y) - u(x)| = |u(\gamma(0)) - u(\gamma(1))| \le \int_0^1 |\nabla u| (\gamma(t)) |\gamma'(t)| \,\mathrm{d}t \,, \tag{2.15}$$

where γ is a geodesic from $\gamma(0) = x$ to $\gamma(1) = y$. Then we average among a sufficiently rich bunch of geodesics. The density estimates allow to turn the average on a bunch of geodesics into an average on balls.

2.2. **Convergence and Gromov's precompactness theorem.** The Gromov-Hausdorff distance is a version of the Hausdorff distance between compact subsets of a given metric space, formulated for general compact metric spaces. See [21, Chapter 7] for an introduction.

Definition 2.13. Let (X_1, d_1) and (X_2, d_2) be compact metric spaces. We consider all the metrics ρ on the disjoint union of X_1 and X_2 that restrict to d_1 and d_2 on X_1 and X_2, respectively. Then we set $\mathsf{d}_{GH}(X_1, X_2)$ to be the infimum among these metrics of all $r > 0$ such that X_1 is contained in the r-enlargement of X_2 and vice-versa.

Remark 2.14. This induces a notion of convergence in the obvious way. It is easy to check that $\mathsf{d}_{GH} = 0$ if and only if the two metric spaces are isometric.

Remark 2.15. X_1 and X_2 are close in the Gromov-Hausdorff distance if they look indistinguishable to the naked eye.

Remark 2.16. In particular, discrete metric spaces are dense in the Gromov Hausdorff topology in the class of compact metric spaces.

All metric spaces will be assumed from now on to be complete and separable.

Remark 2.17. For non compact metric spaces the right notion is pointed Gromov-Hausdorff convergence. We say that a sequence of pointed metric spaces (X_n, d_n, x_n) converge to (X, d, x) in the pointed Gromov-Hausdorff sense provided the balls $B_R(x_n)$ converge in the Gromov Hausdorff sense for any $R > 0$.

There are equivalent ways of characterizing Gromov-Hausdorff convergence.

Definition 2.18. Given metric spaces (X_1, d_1) and (X_2, d_2) and $\varepsilon > 0$ a map $f : X_1 \to X_2$ is an ε-isometry provided

$$|\mathsf{d}_2(f(x_1), f(x_2)) - \mathsf{d}_1(x_1, x_2)| \le \varepsilon \,, \quad \text{for any } x_1, x_2 \in X_1 \,, \tag{2.16}$$

and for any $y \in X_2$ there exists $x \in X_1$ such that

$$\mathsf{d}_2(y, f(x)) \le \varepsilon \,. \tag{2.17}$$

Remark 2.19. We are requiring the map to be almost distance preserving and almost surjective. To the naked eye an ε-isometry looks like an isometry.

Proposition 2.20. *A sequence of compact metric spaces (X_k, d_k) converges to (X, d) in the Gromov-Hausdorff sense if and only if there exist ε_k-isometries $f_k : X_k \to X$ for some sequence $\varepsilon_k \downarrow 0$.*

Idea for total boundedness: given $\varepsilon > 0$ our vision cannot distinguish below the scale ε. Hence we can restrict the metric to $\varepsilon/3$-dense nets and declare equivalent two metrics whose distance functions differ by at most $\varepsilon/3$. Modulo this equivalence the number of possibilities is bounded.

Proposition 2.21. *Given a function $N:\mathbb{R}_+\to\mathbb{N}_+$, the class of all isometry classes of metric spaces with diameter less than $D>0$ and such that for any $\varepsilon>0$ there exists an ε-dense net with at most $N(\varepsilon)$ elements is compact with respect to the Gromov-Hausdorff topology.*

A standard criterion that yields uniform total boundedness is the existence of a doubling measure μ on (X,d).

Definition 2.22. A measure μ on a metric space (X,d) is doubling provided there exists a constant C_d such that

$$\mu(B_{2r}(x))\le C_d\mu(B_r(x))\,,\quad \text{for any } x\in X \text{ and any } r>0\,. \tag{2.18}$$

We fix $\varepsilon>0$ and choose a maximal ε-separated subset $\{x_i\}$. By maximality the set is ε-dense. Moreover the balls $B_{\varepsilon/2}(x_i)$ are disjoint. By iterating the doubling condition we realise that each of the balls $B_{\varepsilon/2}(x_i)$ carries a definite amount of the total mass $\mu(B_D(x))=\mu(X)$.

Bishop-Gromov's inequality Theorem 2.8 and the argument above prove the following precompactness theorem, originally due to Gromov [63] for smooth Riemannian manifolds.

Theorem 2.23. *Given $D>0$, $N\in\mathbb{N}$ and $K\in\mathbb{R}$ the class of $\mathrm{RCD}(K,N)$ metric measure spaces $(X,\mathsf{d},\mathfrak{m})$ with $\mathrm{diam}(X)\le D$ is precompact with respect to the Gromov-Hausdorff topology.*

Example 2.24. A sequence of $2d$ surfaces with nonnegative Gaussian curvature can converge to a cone. Here a singular point appears, in particular regularity is lost. By working a bit harder it is possible to construct sequences of $2d$ surfaces with nonnegative Gaussian curvature GH converging to a metric space with a dense set of conical points, the construction is originally due to Otsu-Shioya [84]. This shows that convergence of higher order is completely unreasonable.

A sequence of flat cylinders with shrinking diameter of the factor $\mathbb{S}^1$ can collapse to a line in the pGH sense. Hence dimension is unstable.

Changes of topology are possible even without drop of the dimension in dimension higher or equal than 4, even though they are much more delicate to construct, the first examples were due to Perelman [87], see also the more recent work of Colding-Naber [41]. We will come back on this later. In particular it is possible to build metrics with positive Ricci on the connected sum of two copies of $\mathbb{CP}^2$ that converge in Gromov-Hausdorff sense without collapse to a singular metric on $\mathbb{S}^4$. Notice that singularities need to appear, this will become more clear later.

Example 2.25. It is possible to attach to a sphere a very thin well and keep the scalar curvature positive. As the well gets thinner and thinner the sequence of manifolds converges Gromov-Hausdorff to a sphere with a segment attached. Notice that the volume of the well is decaying to zero, while the volume of the sphere is not. This is impossible under a lower Ricci bound. Also convergence to limits with different local dimensions is impossible under a lower Ricci bound.

Example 2.26. It is possible to build a sphere with increasingly many thin wells with positive scalar curvature. Ilmanen's example has no GH limit, the sequence converges in the intrinsic flat sense to a standard sphere. If there is only one increasingly thin well, then there is convergence in the GH sense to a sphere with a segment attached. Key issue, when the number of wells goes to infinity there are infinitely many disjoint balls of a radius of a definite size, Bishop Gromov fails, hence there is no Gromov-Hausdorff precompactness.

Example 2.27. Gromov-Lawson and Schoen-Yau constructed model *tunnels* with positive scalar curvature diffeomorphic to $\mathbb{S}^2\times[0,1]$ that attach smoothly on both ends to canonical spheres. They can be made thin and long or thin and short. We can attach spheres with short tunnels to limit to two spheres attached on a point (this phenomenon is called bubbling) or converge to two spheres attached on a segment. It is also possible to attach increasingly many bubbles; in this way there is no Gromov-Hausdorff limit (in particular the uniform doubling estimate fails).

This is impossible under a lower Ricci bound. You can't attach the tunnel without modifying the two components that you are gluing, as a starting point. There is the necessity to scale both the components when we do gluings under a lower Ricci curvature bound.

Remark 2.28. Any n-dimensional smooth Riemannian manifold can be approximated by a graph in the Gromov-Hausdorff sense. Moreover, the graph can be fattened into a nearby n-dimensional smooth Riemannian manifold with positive scalar by replacing the vertexes with small spheres and the edges with even smaller tunnels, see Gromov's lectures [64]. With this construction the topology becomes arbitrarily complex in order to be able to approximate.

Very recently Lee-Topping show that in dimension 4 or higher it is possible to Gromov-Hausdorff approximate any metric on the sphere conformal to the standard one with a sequence of metrics on the sphere with positive scalar curvature, without changing the topology in particular.

Our previous discussion should have convinced that lower Ricci curvature bounds are a matter of combination between volume/reference measure and distance. Hence it is relevant to have a notion of convergence for metric measure spaces. This was first introduced by Fukaya in [51] as a variant of the Gromov-Hausdorff convergence for metric spaces.

In the original definition by Fukaya it is required that the pushforwards of the measures $\mathfrak{m}_n$ via ε_n-isometries f_n weakly converge to $\mathfrak{m}$ as measures on the limit metric space (X, d). There is a natural variant for pointed metric measure spaces where we require the maps to respect the pointed structure and the weak convergence to hold for the restrictions to balls of increasingly radii. It is possible to choose first the approximations depending on the radii, then to make them independent via a diagonal argument.

Several equivalent characterizations are possible for doubling metric measure spaces, here we stick to the following, referring to the work of Gigli-Mondino-Savaré [57] for more on the equivalences.

Definition 2.29. We say that a sequence of pointed metric measure spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converges in the pointed measured Gromov-Hausdorff topology to $(X, \mathsf{d}, \mathfrak{m}, x)$ if there exists a complete and separable metric space (Z, d_Z) and isometric embeddings $\iota_n : X_n \to Z$, $\iota : X \to Z$ such that

$$\int_Z \varphi \, \mathrm{d} \left(\iota_n\right)_\sharp \mathfrak{m}_n \to \int_Z \varphi \, \mathrm{d}\iota_\sharp \mathfrak{m} \, , \quad \text{for any } \varphi \in \mathrm{C_{bs}}(Z) \tag{2.19}$$

and

$$\iota_n(x_n) \to \iota(x) \in \operatorname{supp}\left(\iota_\sharp \mathfrak{m}\right) \, , \tag{2.20}$$

as $n \to \infty$.

Remark 2.30. When there is collapse of the dimension under a lower Ricci bound weights appear in the limit. Moreover, topological boundary might appear too.

Example 2.31. We can consider (N, g_N) and the product $N_\varepsilon := N \times \mathbb{S}^1_\varepsilon$ with metric $g_\varepsilon := g_N + \varepsilon^2 g_{\mathbb{S}^1}$. Then N_ε Gromov-Hausdorff converge to N and measured Gromov-Hasudorff converge (after normalization of the volume) to (N, vol_N).

Example 2.32. Borrowed from [51]. We can consider the ellipsoids

$$x^2 + \frac{(y^2 + z^2)}{\varepsilon^2} = 1 \, . \tag{2.21}$$

As $\varepsilon \downarrow 0$ they converge Gromov-Hausdorff to $[-1,1]$ with standard metric. The normalized volume measures converge to $(1-t^2)^{\frac{1}{2}}\,\mathrm{d}t$ on $[-1,1]$.

Remark 2.33. The compactness theorem is brought to bear on the smooth case as follows. Suppose that we want to prove that a degenerate behaviour is not possible for a certain class of smooth manifolds with curvature bounds. We argue by contradiction. The compactness theorem leads to a limit space exhibiting a particular kind of singularity. If we can show that this degeneracy cannot occur in the limit, then we are done. This is a motivation for the structure theory of limit spaces. More in general, it is one of the motivations for the study of singular spaces with curvature bounds as currently there are statements for limit spaces that cannot be established by taking limits in statements for smooth Riemannian manifolds.

2.3. **Stability of lower Ricci curvature bounds.** The following fundamental stability theorem is the outcome of many contributions. Sturm-Lott-Villani proved stability of the Curvature-Dimension condition (under additional assumptions). Ambrosio-Gigli-Savaré proved the stability of the $\mathrm{BE}(K,N)$ condition under the Sturm-Gromov-Hausdorff convergence in [6]. Gigli-Mondino-Savaré established the stability of the RCD condition under pointed measured Gromov-Hausdorff convergence in [57].

Theorem 2.34. *Let $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$ be pointed $\mathrm{RCD}(K,N)$ metric measure spaces with*

$$\mathfrak{m}_n(B_1(x_n)) \in [c^{-1},c]\,, \tag{2.22}$$

for any $n\in\mathbb{N}$ for some $c>1$, converging in the pointed measured Gromov-Hausdorff topology to $(X,\mathsf{d},\mathfrak{m},x)$. Then $\mathfrak{m}$ is fully supported and $(X,\mathsf{d},\mathfrak{m},x)$ is an $\mathrm{RCD}(K,N)$ space.

Remark 2.35. Analogous conclusions hold when $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x_n)$ are $\mathrm{RCD}(K_n,N_n)$ metric measure spaces for some sequences $K_n\to K\in\mathbb{R}$ and $N_n\to N\in(1,\infty)$.

The following is a remarkable consequence, given that the notion of convergence that we are considering is a very weak one. The expression of the Ricci tensor involves second derivatives of the metric coefficients indeed.

Corollary 2.36. *If a sequence of smooth Riemannian manifolds with dimension N and uniform lower Ricci bounds $\mathrm{Ric}\geq K$ converges in the Gromov-Hausdorff sense to a smooth Riemannian manifold with dimension N, then the lower Ricci bound is maintained.*

Remark 2.37. The above does not require the synthetic theory to be formulated. However, it was completely unclear before its developments.

Remark 2.38. The analogous issues were settled much earlier in the case of lower sectional curvature bounds, thanks to Toponogov's triangle comparison and the theory of Alexandrov spaces with sectional curvature bounded from below.

Exercise 2.39. Find a sequence of smooth Riemannian manifolds with dimension N and $\mathrm{Ric}\geq K$ converging in the Gromov-Hausdorff sense to a k-dimensional manifold (M,g) with $k<N$ such that the bound $\mathrm{Ric}_M\geq K$ is not satisfied.

The key idea for the stability of Curvature-Dimension bounds with respect to (pointed) measured Gromov-Hausdorff convergence is to combine the following:

- the stability of Optimal Transport with respect to Gromov-Hausdorff convergence;
- the lower semicontinuity of non linear functionals on the space of probability measures with respect to weak convergence of both entries, the reference measure and the other entry.

Morally the Lagrangian computations are turned into convexity inequalities and then they are integrated.

Remark 2.40. The infinitesimally Hilbertian requirement is not stable alone. The idea is that it is a first order condition. It becomes stable when coupled with a second order condition, as the synthetic lower Ricci curvature bound.

In particular, the uniform lower Ricci bound can be used to infer certain stability properties of the Cheeger energy under pointed measured Gromov-Hausdorff convergence. Then we can pass to the limit the quadraticity characterizing Riemannian spaces to get that the limit is Riemannian too if the elements of the sequence are Riemannian. See [57].

3. Lecture 3

Doubling and Poincaré alone are enough for developing a basic elliptic regularity theory, Harnack's inequality and Hölder continuity for harmonic functions, for instance, see [15]. They are also enough for constructing a differentiable structure on metric measure spaces, see the seminal work of Cheeger [25] for instance.
Now we shall see that the structure of $\mathrm{RCD}(K,N)$ spaces is much more rich. The starting point of our analysis are a series of rigidity theorems. Most of them might fail for general $\mathrm{CD}(K,N)$ spaces in the present form.

Remark 3.1. Under a positive lower Ricci bound certain geometric objects are shown not to exist. Example: Bonnet-Meyers diameter bounds, $\mathrm{RCD}(N-1,N)$ spaces have finite diameter. Under the corresponding weak inequality, these objects might exist, but only in presence of special geometric structure, cf. with Cheeger's lectures [26].

Remark 3.2. There is a distinction between settings where the rigid model situation is unique, in which case we talk about *stability*, and settings where the models are a whole family of spaces, in which case we talk about *almost rigidity*. Cf. with the discussion in the introduction of [29].

We will be concerned (mainly) with two geometric rigidities: cone structures and splittings. In both cases, the special geometric structure is that of a *warped product*.

Definition 3.3. In the manifold case, warped products are Riemannian metrics on products that can be written as $g = \mathrm{d}r^2 + k^2(r)\tilde{g}$, where $\tilde{g}$ denotes a Riemannian metric on some manifold $\tilde{M}^{n-1}$ called the *cross section*.

For line splittings, $k \equiv 1$ on $\mathbb{R}$; for metric cones, $k(r) = r$ on $(0,\infty)$. It is important that they admit generalizations to the setting of metric (measure) spaces.

Definition 3.4. Given metric measure spaces $(X, \mathsf{d}_X, \mathfrak{m}_X)$ and $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ we can consider the product structure $X \times Y$ with

$$\mathsf{d}^2_{X\times Y}((x_1,y_1),(x_2,y_2)) := \mathsf{d}^2_X(x_1,x_2) + \mathsf{d}^2_Y(y_1,y_2)\,, \tag{3.1}$$

$$\mathfrak{m}_{X\times Y} = \mathfrak{m}_X \otimes \mathfrak{m}_Y\,, \tag{3.2}$$

where $\mathfrak{m}_X \otimes \mathfrak{m}_Y(A\times B) := \mathfrak{m}_X(A)\cdot\mathfrak{m}_Y(B)$ for any Borel sets $A \subset X$ and $B \subset Y$.

3.1. **Splitting theorem and linear functions.**

The starting point is the observation that convex subsets of $\mathbb{R}^n$ that contain a line must be cylinders, i.e. they split as the product of $\mathbb{R}$ with a convex subset of $\mathbb{R}^{n-1}$.

3.1.1. *The Cheeger-Gromoll splitting theorem.*

Definition 3.5. A ray $\gamma : [0,\infty) \to X$ is a curve which is minimizing between any two of its points. A line is a curve $\gamma : \mathbb{R} \to X$ which is minimizing between any two of its points.

The statement for smooth Riemannian manifolds is due to Cheeger-Gromoll [33].

Theorem 3.6. *Let* (M,g) *be a smooth Riemannian manifold with* $\mathrm{Ric} \geq 0$*. If* M *contains a line, then it splits isometrically as* $M = N \times \mathbb{R}$ *where* (N,h) *is an* $(n-1)$*-dimensional Riemannian manifold with* $\mathrm{Ric} \geq 0$.

Remark 3.7. The analogous statement was proved earlier by Toponogov for nonnegative sectional curvature.

For Ricci limit spaces the splitting theorem is due to Cheeger-Colding [29] and it required a variety of new techniques.

Theorem 3.8. *If (X, d) is a Gromov-Hausdorff limit of Riemannian manifolds (M_i, g_i) with $\mathrm{Ric}_i \geq -\varepsilon_i$, $\varepsilon_i \to 0$ and (X, d) contains a line, then it splits isometrically and the limit measure is a product measure.*

Remark 3.9. Cheeger-Colding needed to prove that a smooth manifold with almost nonnegative Ricci curvature almost containing a line is close in the Gromov-Hausdorff sense to a product by producing explicitly a cross section and a Gromov-Hausdorff approximation. Cf. with the recent work of Xu [97] for a quantitative estimate with explicit dependence on the parameters.

For $\mathrm{RCD}(0, N)$ spaces the splitting theorem is due to Gigli. Some key ingredients are the same but new steps are needed, fundamentally due to the absence of a second order differential calculus for RCD spaces at the time of [53].

Theorem 3.10. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ metric measure space for some $1 \leq N < \infty$. Assume that it contains a line, then it splits as $X = \mathbb{R} \times Y$, where $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ is an $\mathrm{RCD}(0, N-1)$ metric measure space.*

Remark 3.11. The requirement that $N < \infty$ is fundamental.

We outline a proof, with current technology, much closer to the original one for smooth Riemannian manifolds.

Remark 3.12. Linear functions are exactly those which are simultaneously harmonic and distance functions on $\mathbb{R}^n$. Notice that the affine condition in this way does not require the Hessian, rather it involves the Laplacian and the (modulus of the) gradient.

Definition 3.13. The Busemann function b_γ of a ray $\gamma : [0, \infty) \to X$ is defined as

$$b_\gamma(x) := \lim_{s \to \infty} \left(\mathsf{d}(x, \gamma(s)) - s\right) . \tag{3.3}$$

Notice that the function inside the parenthesis is monotone nonincreasing and bounded from below by $-\mathsf{d}(x, \gamma(0))$ by the triangle inequality.

Remark 3.14. Intuition: in the end we want the Busemann function to be affine and in the direction of the splitting. This is what happens in $\mathbb{R}^n$ and, more in general, for any product $\mathbb{R} \times X$.

Lemma 3.15. *The Busemann function is superharmonic.*

Proof. Apply the Laplacian comparison Theorem 2.5 to get

$$\Delta \mathsf{d}_{\gamma(s)} \leq \frac{N-1}{\mathsf{d}(\cdot, \gamma(s))} . \tag{3.4}$$

As $s \to \infty$ the denominator goes to $+\infty$. Hence we formally get

$$\Delta b_\gamma \leq 0 . \tag{3.5}$$

□

Remark 3.16. The above does not require the infinitesimally Hilbertian assumption.

Now we consider a line γ and view it as gluing two rays, γ^+ and γ^-. By triangle inequality

$$b_{\gamma^+} + b_{\gamma^-} \geq 0 , \tag{3.6}$$

with equality exactly along γ.

Remark 3.17. In the first proof the key issue to be circumvented was the a priori non regularity of the Busemann function. This required superharmonicity to be understood in weak sense.

Lemma 3.18. $b_{\gamma^+} = -b_{\gamma^-}$ *and they are both harmonic functions.*

Proof. Notice that $b_{\gamma^+} + b_{\gamma^-}$ is superharmonic by linearity of the Laplacian and attains a minimum along γ. Hence it is constant by the strong maximum principle[1]. We conclude that the identity holds and $\Delta b_{\gamma^+} = 0$, i.e. the Busemann function is harmonic. □

Lemma 3.19. $|\nabla b_\gamma| = 1$ *everywhere.*

Proof. It is elementary to check that b_γ is 1-Lipschitz, hence $|\nabla b| \le 1$. In order to prove the converse inequality we check that the 1-Lipschitz inequality is saturated by finding points for which

$$|b_\gamma(x) - b_\gamma(y)| = \mathsf{d}(x,y)\,. \tag{3.7}$$

□

The idea then is to employ Bochner's identity to obtain

$$0 = \frac{1}{2}\Delta\,|\nabla b|^2 = \|\mathrm{Hess}\, b\|^2 + \nabla b \cdot \nabla \Delta b + \mathrm{Ric}(\nabla b, \nabla b) = \|\mathrm{Hess}\, b\|^2 + \mathrm{Ric}(\nabla b, \nabla b)\,. \tag{3.8}$$

Hence

$$\mathrm{Hess}\, b = 0\,. \tag{3.9}$$

Combined with $\Delta b = 0$, this means that b flows by measure preserving isometries. Moreover, it induces a splitting. The space is the product of a level set of the Busemann function (which is totally geodesic) with the standard $\mathbb{R}$. Setting X_t its flow, we can check that the map:

$$\Phi : \{b = 0\} \times \mathbb{R} \to X\,, \quad \Phi(x,t) := X_t(x) \tag{3.10}$$

is an isometry. Indeed

$$\mathsf{d}^2(\Phi(x,t), \Phi(y,s)) = \mathsf{d}^2(X_t(x), X_s(y)) = \mathsf{d}^2(x, X_{s-t}(y)) = |s-t|^2 + \mathsf{d}^2(x,y)\,. \tag{3.11}$$

The latter equality can be checked by computing

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathsf{d}^2(x, X_r(y))|_{r=0} = 0\,, \tag{3.12}$$

taking into account that the level set is totally geodesic, and

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathsf{d}^2(x, X_r(x)) = 2\,, \quad \text{for any } r \in \mathbb{R}\,, \tag{3.13}$$

using the second variation formula along geodesics.

On an RCD space this is far from being clear. There are two issues: how to obtain a Bochner identity/inequality with Hessian term and the proof of the Pythagorean identity. I will focus on the first one. See [17, Theorem 3.4] for the a detailed proof building on top of the second order differentiation formular obtained by Gigli-Tamaninini in [60].

[1] A superharmonic function attaining a minimum at an interior point is constant; this is originally due to E. Hopf in the Euclidean setting, back in 1927. Notice that the statement holds even under the weaker doubling and Poincaré assumptions.

3.1.2. *Self-improvement of the Curvature-Dimension condition.* A fundamental idea originally due to Bakry [12] is that the Curvature-Dimension condition, formulated in terms of Γ-calculus as in Section 1.2.2, admits a self improvement. We briefly illustrate the argument neglecting all the regularity issues, in the smooth setting. Recall that the curvature dimension condition can be formulated as

$$\Gamma_2(f) \geq K\Gamma(f)\,, \tag{3.14}$$

where

$$\Gamma(f) = |\nabla f|^2\,, \quad \Gamma_2(f) = \frac{1}{2}\Delta\,|\nabla f|^2 - \nabla f \cdot \nabla \Delta f\,. \tag{3.15}$$

The idea is to apply the Γ_2 inequality to suitably chosen auxiliary functions, employing standard differential calculus rules. From the calculus rules in Riemannian geometry, for any sufficiently regular functions f, g, h it holds

$$\mathrm{Hess}(f)(\nabla g, \nabla h) = \frac{1}{2}\left[\Gamma(g, \Gamma(f,h)) + \Gamma(h, \Gamma(f,g)) - \Gamma(f, \Gamma(g,h))\right]\,. \tag{3.16}$$

We are going to prove that for sufficiently regular functions f_1, f_2, f_3

$$(\mathrm{Hess}(f_1)(\nabla f_3, \nabla f_3))^2 \leq \left[\Gamma_2(f_1) - K\Gamma(f_1)\right]\Gamma(f_2)\Gamma(f_3)\,, \tag{3.17}$$

which is a simpler variant of the inequality

$$\Delta\frac{1}{2}\,|\nabla f|^2 \geq \|\mathrm{Hess}\, f\|^2_{\mathrm{HS}} + \nabla f \cdot \nabla \Delta f + K\,|\nabla f|^2\,, \tag{3.18}$$

that has been proved in [55]. See [58, Proposition 6.2.20] for a similar discussion.

We consider $f := (f_1, \dots, f_k) : X \to \mathbb{R}^k$ and a smooth function $\Psi : \mathbb{R}^k \to \mathbb{R}$. By the calculus rules we can compute

$$\Gamma_2(\Psi(f)) = \sum_{i,j=1}^k X_i X_j \Gamma_2(f_i, f_j) + \sum_{i,j,l=1}^k X_i Y_{jl}\,\mathrm{Hess}(f_i)(\nabla f_j, \nabla f_l) \tag{3.19}$$

$$+ \sum_{i,j,l,m=1}^k Y_{ij} Y_{lm} \Gamma(f_i, f_l)\Gamma(f_j, f_m)\,, \tag{3.20}$$

where

$$X_i = \partial_i \Psi(f_1, \dots, f_k)\,, \quad Y_{ij} = \partial^2_{ij}\Psi(f_1, \dots, f_k)\,. \tag{3.21}$$

At any point x the function $\Psi : \mathbb{R}^k \to \mathbb{R}$ can be chosen in such a way that the coefficients X_i and Y_{jl} take any particular value provided the symmetries $Y_{jl} = Y_{lj}$ are respected (quadratic polynomials).

Then the Curvature-Dimension condition (3.14) provides a nonnegative quadratic form in the variables (X_i, Y_{jl}). We limit to the case of three functions (f_1, f_2, f_3) and restrict the quadratic form to the set where all variables are 0 except from X_1 and Y_{23}. The determinant of the quadratic form is

$$\left[\Gamma_2(f_1) - K\Gamma(f_1)\right]\left[\Gamma(f_2, f_3)^2 + \Gamma(f_2)\Gamma(f_3)\right] - 2\left(\mathrm{Hess}(f_1)(\nabla f_2, \nabla f_3)\right)^2\,. \tag{3.22}$$

The positivity of (3.22) proves (3.17).

Remark 3.20. In the setting of $\mathrm{RCD}(K,\infty)$ metric measure spaces, the above is technically much more demanding. In [91] Savaré extended Bakry's strategy to the setting of Dirichlet forms verifying the $\mathrm{BE}(K,\infty)$ condition, obtaining the sought control on the right hand side of (3.16). Later in [55] Gigli developed a second order differential calculus on RCD spaces where a notion of Hessian can be introduced by integration by parts, and it is consistent with (3.16).

With the second order calculus and the self improvement of the RCD condition, it is possible to prove that the Busemann function of a line is affine on $\mathrm{RCD}(0,N)$ spaces as in the original proof by Cheeger-Colding.

Remark 3.21. The splitting theorem can be iterated to gain additional splittings. This is the starting point for the regularity theory.

Corollary 3.22. *Let* $(\mathbb{R}^n, \mathsf{d}_{\rm eucl}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0,N)$ *metric measure space and assume that* $\mathfrak{m}$ *is fully supported. Then* $\mathfrak{m} = \mathscr{L}^n$*, up to multiplicative constants.*

Proof. As (X,d) is isometric to $\mathbb{R}^n$, there exists a line. By the splitting Theorem 3.10, $X = \mathbb{R}\times X_1$ with the product metric measure space structure, where $(X_1,\mathsf{d}_1,\mathfrak{m}_1)$ is an $\mathrm{RCD}(0,N-1)$ metric measure space. As (X,d) is isometric to $\mathbb{R}^n$, (X_1,d_1) is isometric to $\mathbb{R}^{n-1}$, with canonical Euclidean metric. Then we can iterate the previous construction. □

3.2. **Almost splitting theorem.** After proving the splitting theorem in the non smooth context, Gromov's precompactness theorem and the stability of the RCD condition Theorem 2.34 easily yield the following.

Theorem 3.23. *There exists a positive function* $(\delta, L, \varepsilon, N, R)\mapsto \Psi(\delta, L,\varepsilon|N,R)$ *such that for given* $N\ge 1$ *and* $R>0$ *it holds*

$$\lim_{\varepsilon,\delta,L^{-1}\to 0}\Psi(\delta,L,\varepsilon|N,R)=0 \tag{3.23}$$

and the following is true. If $(X,\mathsf{d},\mathfrak{m})$ *is an* $\mathrm{RCD}(-\delta,N)$ *metric measure space,* $x,y_1,y_2\in X$ *are such that* $\min\{\mathsf{d}(x,y_1),\mathsf{d}(x,y_2)\}\ge L$ *and*

$$\mathsf{d}(x,y_1)+\mathsf{d}(x,y_2)-\varepsilon\le \mathsf{d}(y_1,y_2)\,, \tag{3.24}$$

then there exist an $\mathrm{RCD}(0,N-1)$ *metric measure space* $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ *and a point* $y\in Y$ *such that*

$$\mathsf{d}_{\rm mGH}\left(B_R(x), B_R^{\mathbb{R}\times Y}((0,y))\right)\le \Psi(\delta,\varepsilon,L|N,R)\,. \tag{3.25}$$

Proof. Assume by contradiction that there is no such function ψ. We find a sequence of $\mathrm{RCD}(-1/n,N)$ spaces $(X_n,\mathsf{d}_n,\mathfrak{m}_n)$ and points x^n, y_1^n, y_2^n such that

$$\min\{\mathsf{d}(x^n,y_1^n),\mathsf{d}(x^n,y_2^n)\}\ge n\,,\quad \mathsf{d}(x^n,y_1)+\mathsf{d}(x,y_2^n)-1/n\le \mathsf{d}(y_1^n,y_2^n) \tag{3.26}$$

and it holds

$$\mathsf{d}_{mGH}\left(B_R^{X_n}(x^n), B_R^{\mathbb{R}\times Y}((0,y))\right)>\varepsilon_0>0\,, \tag{3.27}$$

for any $\mathrm{RCD}(0,N-1)$ metric measure space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ and any $n\in\mathbb{N}$, $n\ge 1$.

Without loss of generality we assume that $\mathfrak{m}_n(B_R(x^n)) = 1$ for any $n\in\mathbb{N}$. By Gromov's precompactness theorem, up to the extraction of a subsequence that we do not relabel, $(X_n,\mathsf{d}_n,\mathfrak{m}_n,x^n)$ converge in the pointed measured Gromov-Hausdorff topology to an $\mathrm{RCD}(0,N)$ metric measure space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. Taking the limit as $n\to\infty$ in (3.26) we find that (Z,d_Z) contains a line. By the splitting Theorem 3.10, $Z=\mathbb{R}\times Y$ for some $\mathrm{RCD}(0,N-1)$ metric measure space $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$. This is in contradiction with (3.27) for n large enough. □

4. Lecture 4

4.1. **Metric measure cones and Ricci lower bounds.** The cone warped product metric $g=\mathrm{d}r^2+r^2\tilde g$ admits a metric characterization.

Definition 4.1. Given a metric space (X,d_X) the metric cone $\left(C(X),\mathsf{d}_{C(X)}\right)$ is defined as $[0,\infty)\times X$ with the distance

$$\mathsf{d}^2_{C(X)}\left((r_1,x_1),(r_2,x_2)\right)=r_1^2+r_2^2-2r_1r_2\cos(\min\{\mathsf{d}_X(x_1,x_2),\pi\})\,. \tag{4.1}$$

Given a metric measure space $(X,\mathsf{d}_X,\mathfrak{m}_X)$, the N-metric measure cone is the metric measure space $(C(X),\mathsf{d}_{C(X)},\mathfrak{m}_{C(X)})$ where

$$\mathfrak{m}_{C(X)}=r^{N-1}\,\mathrm{d}r\otimes\mathfrak{m}_X\,. \tag{4.2}$$

Exercise 4.2. If (X, d) is isometric to a smooth Riemannian manifold, then check that the metric cone over X is a smooth Riemannian manifold away from a singular point. Moreover, the Ricci curvature of the cone is nonnegative away from the singular point if and only if the Ricci curvature of the cross section satisfies $\mathrm{Ric}_X \geq (N-2)g_X$, where $(N-1)$ is the dimension of the section X.

Check that the cone $C(X)$ is Ricci-flat away from the singular point if and only if the cross section is Einstein, i.e. $\mathrm{Ric}_X = (N-2)g_X$.

Exercise 4.3. Re-check that the usual calculus rules on warped products hold on cones. Try to find a *synthetic* proof. For instance, check the expression for the Laplacian in coordinates (r, x),

$$\Delta_{C(X)} = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \Delta_X \,, \tag{4.3}$$

by using the integration by parts formula in Definition 1.27.

Of course the singularity at the origin is an issue for the classical Riemannian theory. The RCD framework is very well suited to include the singularity in the theory, which is fundamental for certain applications (analysis on blow-downs of open manifolds with nonnegative Ricci and Euclidean volume growth, for instance, cf. with Corollary 4.32).

We first need to digress on spectral gap estimates for the Laplacian on $\mathrm{RCD}(N-1, N)$ spaces $(X, \mathsf{d}, \mathfrak{m})$. It is a classical result, due to Lichnerowicz, that the smallest eigenvalue of the Laplacian on a closed Riemannian manifold (M^N, g) with $\mathrm{Ric} \geq N-1$ is larger or equal than N.

Definition 4.4. Given a compact $\mathrm{RCD}(K, N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ we define the first eigenvalue of the Laplacian as

$$\lambda_1\left((X, \mathsf{d}, \mathfrak{m})\right) = \lambda_1 := \inf\left\{ \frac{\int_X |\nabla f|^2 \,\mathrm{d}\mathfrak{m}}{\int_X f^2 \,\mathrm{d}\mathfrak{m}} \;:\; f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \,,\, f \neq 0 \,,\, \int_X f \,\mathrm{d}\mathfrak{m} = 0 \right\} .$$

Remark 4.5. On closed Riemannian manifolds it is a classical fact that the variational problem above characterizes the smallest eigenvalue of the Laplacian, which is a densely defined, self adjoint linear operator on $L^2(M, \mathrm{vol})$ with discrete spectrum. The variational principle is a special case of the more general characterization of eigenvalues usually known as Courant-Fischer-Weyl min-max principle.

Theorem 4.6. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *metric measure space. Then* $\lambda_1 \geq N$.

Proof. We present a formal proof, see [47, Theorem 4.2] for a rigorous one along the same lines.

Let f be an eigenfunction associated to the first eigenvalue of the Laplacian $\Delta f = -\lambda_1 f$ and assume it is normalized to $\int f^2 \,\mathrm{d}\mathfrak{m} = 1$. The existence of a minimizer for the variational problem in Definition 4.4 follows from standard functional analytic arguments based on the compactness (which follows from the Bonnet-Meyers theorem), the Poincaré inequality Theorem 2.11 and a Sobolev inequality (see for instance [95, Theorem 30.23]). In turn, these ingredients imply that the embedding of $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ into $L^2(X, \mathfrak{m})$ is compact, see for instance [65]. The fact that a minimizer of the variational problem is a solution of $\Delta f = -\lambda_1 f$ can be proved arguing as in the classical Riemannian case, see for instance [56].

Bochner's inequality gives

$$\Delta \frac{1}{2} |\nabla f|^2 \geq \frac{(\Delta f)^2}{N} + \nabla f \cdot \nabla \Delta f + (N-1) |\nabla f|^2 \,. \tag{4.4}$$

Plugging in $\Delta f = -\lambda_1 f$, we get

$$\Delta \frac{1}{2} |\nabla f|^2 \geq \frac{\lambda_1^2}{N} - \lambda_1 |\nabla f|^2 + (N-1) |\nabla f|^2 \, . \tag{4.5}$$

Integrating (4.5) on X with respect to $\mathfrak{m}$ we get

$$0 \geq \frac{\lambda_1^2}{N} - \lambda_1^2 + (N-1)\lambda_1 \, . \tag{4.6}$$

The sought estimate follows. □

Remark 4.7. The estimate is sharp, as equality is attained on the sphere with canonical Riemannian metric. This is the only case of equality in the Riemannian case, the original proof is due to Obata [83]. In the setting of $\mathrm{RCD}(N-1, N)$ metric measure spaces the equality cases are rigid. However, the equality $\lambda_1 = N$ only implies isomorphism with a spherical suspension [73], which is a generalization to the setting of metric measure spaces of the warped product metric

$$\bar{g} = \mathrm{d}r^2 + \sin^2 r g_M \, , \quad \text{on } [0, \pi] \times M \, . \tag{4.7}$$

The following is due to Ketterer [74].

Theorem 4.8. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then the metric measure cone*

$$\Big(C(X), \mathsf{d}_{C(X)}, \mathfrak{m}_{C(X)} \Big) \tag{4.8}$$

is an $\mathrm{RCD}(0, N)$ *metric measure space if and only if the cross section is an* $\mathrm{RCD}(N-2, N-1)$ *metric measure space.*

Proof. The idea is to prove that the $\mathrm{RCD}(0, N)$ condition of the cone and the $\mathrm{RCD}(N-2, N-1)$ condition for the cross section are equivalent through the Eulerian approach based on Bochner's inequality.
There are two main steps. The first one is proving that the Γ_2 inequality on an algebra of tensor product functions is equivalent to the Γ_2 inequality on the cross section. This is proved by explicit computation, by relating the differential objects of the cone to the differential operators on the cross section.
The second step is proving that this algebra is dense enough to establish the full Γ_2 inequality required to check that RCD condition. This is needed. The elementary example of a circle over a cone with too large diameter shows that the right Bochner inequality on the cross section is not sufficient for establishing the RCD condition on the cone. Here the spectral gap inequality Theorem 4.6 plays a key role. □

4.2. **Volume cone implies metric cone.** As for the splitting theorem, metric measure cones are characterized by the existence of a rigid function. In the case of the splitting theorem, the rigid function is affine, i.e. it has vanishing Hessian. For cones with smooth cross section and vertex p, $r^2 := \mathsf{d}_p^2$ verifies

$$\mathrm{Hess}\, r^2 = 2g \, , \quad \Delta r^2 = 2N \, , \quad \left| \nabla r^2 \right| = 2r^2 \, , \tag{4.9}$$

where g denotes the Riemannian metric of the cone (outside the vertex). Notice that the Laplacian identity follows from the Hessian identity by tracing. However, it is relevant to keep them separated, as for weighted Riemannian manifolds the Laplacian is not the trace of the Hessian, in general, and the proof will proceed the other way around. Namely, the gradient and Laplacian identities imply the Hessian identity (if the Ricci curvature is nonnegative).

Remark 4.9. This is completely analogous to the splitting theorem: a harmonic function with constant norm of the gradient is affine if the Ricci curvature is nonnegative.

The following volume cone to metric cone theorem for RCD spaces is due to De Philippis-Gigli [44].

Theorem 4.10. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ metric measure space. Let $p \in X$, $0 < r < R$ be such that*

$$\mathfrak{m}(B_R(p)) = \left(\frac{R}{r}\right)^N \mathfrak{m}(B_r(p))\,. \tag{4.10}$$

Then one of the following three possibilities occurs:

i) *the sphere $S_{R/2}(p)$ contains exactly one point. Then (X, d) is isometric to a 1-manifold with boundary and p is a boundary point;*
ii) *the sphere $S_{R/2}(p)$ contains exactly two points. Then (X, d) is isometric to a 1-manifold (possibly with boundary) and p is an interior point;*
iii) *the sphere $S_{R/2}(p)$ contains at least three points. Then $N \geq 2$ and there exists an $\mathrm{RCD}(N-2, N-1)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ such that the ball $B_R(p)$ is locally isometric to the ball centred at the tip of the metric measure cone $C(Z)$, Moreover, the local isometry is an isometry restricted to $B_{R/2}(p)$.*

Proof. We outline the main points of the strategy, neglecting the technical issues. With the recent developments of the RCD theory this outline can be made rigorous. The original proof by De Philippis-Gigli in [44] was much more delicate.

There are three main steps: proving that under (4.10) $\Delta \mathsf{d}_p^2 = 2N$ on $B_R(p)$; deducing that $\mathrm{Hess}\,\mathsf{d}_p^2 = 2g$ on $B_R(p)$; building the isometry with the cone.

Notice that (4.10) corresponds to the equality case in Bishop-Gromov's inequality. By the way we proved Bishop-Gromov after (2.10), this shows that equality is attained in the Laplacian comparison for d_p^2. Namely,

$$\Delta \mathsf{d}_p^2 = 2N \quad \text{on } B_R(p)\,. \tag{4.11}$$

Notice that $|\nabla \mathsf{d}_p| = 1$. Hence by chain rule $\left|\nabla \mathsf{d}_p^2\right| = 2\mathsf{d}_p^2$.

Applying Bochner's inequality with Hessian term to $r^2 = \mathsf{d}_p^2$ we obtain:

$$\Delta \frac{1}{2}\left|\nabla r^2\right|^2 \geq \left\|\mathrm{Hess}\, r^2\right\|^2 \geq \frac{\left(\Delta r^2\right)^2}{N}\,. \tag{4.12}$$

At the left hand side

$$\Delta \frac{1}{2}\left|\nabla r^2\right|^2 = \Delta\left(2r^2\right) = 4N\,. \tag{4.13}$$

At the right hand side

$$\frac{\left(\Delta r^2\right)^2}{N} = 4N\,. \tag{4.14}$$

Hence equality holds in (4.12):

$$\left\|\mathrm{Hess}\, r^2\right\|^2 = \frac{\left(\Delta r^2\right)^2}{N}\,. \tag{4.15}$$

For the sake of completing the argument, we assume that the Laplacian equals the trace of the Hessian, but this is not needed. If this is the case, denoting by $\lambda_1, \cdot, \lambda_N$ the eigenvalues of $\mathrm{Hess}\, r^2$, it holds

$$\left\|\mathrm{Hess}\, r^2\right\|^2 = \sum_{i=1}^N \lambda_1^2\,, \quad \Delta r^2 = \sum_{i=1}^N \lambda_i\,. \tag{4.16}$$

By (4.15), equality in the Cauchy-Schwarz inequality holds. Hence, taking into account that $\Delta r^2 = 2N$, we infer that all the eigenvalues of $\mathrm{Hess}\, \frac{1}{2} r^2$ equal 1. Therefore

$$\mathrm{Hess}\, \frac{1}{2}\mathsf{d}_p^2 = g\,, \quad \text{on } B_R(p)\,. \tag{4.17}$$

The Hessian identity (4.17) implies that the gradient flow of the vector field $\frac{1}{2}\nabla \mathsf{d}_p^2$ is by homotheties, cf. with the discussion after (1.12). Moreover it induces a cone warped product structure.

For any $0 < r < R/2$, let us consider the sphere

$$S_r(p) := \{x \in X \,:\, \mathsf{d}(x,p) = r\} \tag{4.18}$$

and endow it with the distance induced as

$$\mathsf{d}_r'(x,y)^2 := \inf\left\{\int_0^1 |\gamma_t'|^2\,\mathrm{d}t\right\}, \tag{4.19}$$

with the infimum running among all Lipschitz curves $\gamma : [0,1] \to S_r(p)$ with $\gamma(0) = x'$ and $\gamma(1) = y'$. Above, the speed is computed with respect to the metric space (X, d). In the smooth Riemannian setting, d_r' would be the distance induced by the Riemannian metric obtained by restriction of the ambient Riemannian metric to the submanifold.

Moreover by the coarea formula, letting Per_r be the perimeter measure of $B_r(p)$, it holds

$$\int_{B_R(p)} \varphi\,\mathrm{d}\mathfrak{m} = \int_0^R \int_{S_r} \varphi\,\mathrm{d}\left(\frac{1}{r^{N-1}}\mathrm{Per}_r\right) r^{N-1}\,\mathrm{d}r\,, \tag{4.20}$$

for any continuous function $\varphi : X \to \mathbb{R}$ with compact support. We set $\mathfrak{m}_r := \mathrm{Per}_r / r^{N-1}$.

We denote by X_t the gradient flow of $\frac{1}{2}\nabla \mathsf{d}_p^2$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t(x) = -\left(\nabla \frac{1}{2}\mathsf{d}_p^2\right)(X_t(x))\,. \tag{4.21}$$

Then the following are satisfied:

i) X_t is a homotethy between $(S_{R/2}, \mathsf{d}')$ and $(S_{e^{-t}R/2}, \mathsf{d}')$ for any $t \geq 0$;

ii)
$$(X_t)_\sharp\, \mathfrak{m}_{R/2} = \mathfrak{m}_{e^{-t}R/2}\,, \quad \text{for any } t \geq 0\,; \tag{4.22}$$

iii) setting $(Z, \mathsf{d}_Z, \mathfrak{m}_Z) := (S_{R/2}(p), 2\mathsf{d}'_{R/2}/R, \mathfrak{m}_{R/2})$, it holds that $B_{R/2}(p)$ is isomorphic to the ball centred at the tip with radius $R/2$ in the cone $C(Z)$. The proof of this fact builds on top of the second order differentiation formula from [60].

Applying Theorem 4.8 we infer that $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ is an $\mathrm{RCD}(N-2, N-1)$ metric measure space. □

Exercise 4.11. Check that $C(Z)$ contains lines if and only if $\mathrm{diam}(Z) = \pi$. Hence, by the splitting Theorem 3.10, $C(Z)$ splits a line if and only if the diameter of the cross section is maximal. By the characterization of rigidity for the spectral gap Theorem 4.6, this is equivalent to $\lambda_1(Z) = N - 1$.

Remark 4.12. For (pointed) measured Gromov-Hausdorff limits of smooth manifolds with almost nonnegative Ricci curvature the above was proved earlier by Cheeger-Colding [29] and, again, it required explicit estimates of the Gromov-Hausdorff distance from a cone metric. Basically no information was obtained on the cross section, besides it being a length metric space.

4.3. **N-dimensional $\mathrm{RCD}(K,N)$ spaces.** The focus for the rest of the course will be on those $\mathrm{RCD}(K,N)$ metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ for which $\mathfrak{m} = \mathscr{H}^N$, the N-dimensional Hausdorff measure on the metric space (X, d).

We recall some basic terminology about Hausdorff measures, see for instance [10, Chapter 2] or the classical [48].

Definition 4.13. Let (X, d) be a metric space. For $k \geq 0$ we let

$$\omega_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)} . \tag{4.23}$$

If $\delta \in (0, \infty)$ and $E \subset X$ we let

$$\mathscr{H}^k_\delta(E) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{i \in I} (\mathrm{diam} E_i)^k \ : \ E \subset \cup_{i \in I} E_i \, , \ \mathrm{diam}(E_i) \leq \delta \right\} \tag{4.24}$$

be the k-dimensional pre-Hausdorff measure of E.
Moreover, we let

$$\mathscr{H}^k(E) := \sup_{\delta > 0} \mathcal{H}^k_\delta(E) \tag{4.25}$$

be the k-dimensional Hausdorff measure of E.

Remark 4.14. It is easily verified that $\delta \mapsto \mathcal{H}^k_\delta(E)$ is a nonincreasing function. Hence the supremum in (4.25) can be replaced with a limit as $\delta \to 0$.

Exercise 4.15. If (X, d) is isometric to a smooth Riemannian manifold of dimension n, then $\mathscr{H}^n$ coincides with the Riemannian volume measure vol.

Basic properties of the Hausdorff (pre-)measures are the following:

i) for all $k \geq 0$ and all $\delta \in (0, \infty)$, $\mathscr{H}^k_\delta$ and $\mathscr{H}^k$ are outer measures;
ii) for all $k \geq 0$, $\mathscr{H}^k$ is a Borel measure;
iii) for any $E \subset X$ and any $k \geq 0$,

$$\mathscr{H}^k(E) > 0 \Rightarrow \mathscr{H}^{k'}(E) = +\infty \, , \quad \text{for any } 0 \leq k' < k \, . \tag{4.26}$$

Then the definition of Hausdorff dimension for (subsets of) metric spaces can be given.

Definition 4.16. Given a subset E of a metric space (X, d) we let

$$\dim_{\mathscr{H}}(E) := \inf \left\{ k \geq 0 \ : \ \mathscr{H}^k(B) = 0 \right\} . \tag{4.27}$$

There is a notion of density with respect to the Hausdorff measure.

Definition 4.17. Let (X, d) be a complete and separable metric space and let $k \geq 0$. Let $\mathfrak{m}$ be a Radon measure on X. We define the k-upper density function of $\mathfrak{m}$ as

$$\overline{\theta}_k(\mathfrak{m}, x) := \limsup_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{\omega_k r^k} . \tag{4.28}$$

We refer for instance to [10, Theorem 2.4.3] for a proof of the following well known criterion.

Lemma 4.18. *For every Borel set $E \subset X$ the following hold:*

i) *if $\overline{\theta}_k(\mathfrak{m}, x) \geq c$ for every $x \in E$, then $\mathfrak{m}(E) \geq c\mathscr{H}^k(E)$;*
ii) *if $\overline{\theta}_k(\mathfrak{m}, x) \leq c$ for every $x \in E$, then $\mathfrak{m}(E) \leq 2^k c\mathscr{H}^k(E)$.*

Corollary 4.19. *If $\mathscr{H}^k(E) < \infty$ for some $E \subset X$, then*

$$\limsup_{r \to 0} \frac{\mathscr{H}^k(E \cap B_r(x))}{\omega_k r^k} \leq 1 \, , \quad \textit{for } \mathscr{H}^k\textit{-a.e. } x \in E \, . \tag{4.29}$$

Remark 4.20. For a $\mathrm{CD}(K, N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ the Hausdorff dimension of (X, d) is always less or equal than N. This follows from the Bishop-Gromov monotonicity Theorem 2.8 and Lemma 4.18. Indeed, for any $K \in \mathbb{R}$, $1 \leq N < \infty$ and $R > 0$ there exists a constant $C(K, N, R) > 0$ such that

$$\mathscr{H}^N \llcorner B_R(x) \leq C(K, N, R) \frac{\mathfrak{m} \llcorner B_R(x)}{\mathfrak{m}(B_1(x))} , \tag{4.30}$$

for any $x \in X$. In particular, $\mathscr{H}^N$ is absolutely continuous with respect to $\mathfrak{m}$ and it does not charge the spheres:

$$\mathscr{H}^N(B_r(x)) = \mathscr{H}^N(\overline{B}_r(x)) \, , \quad \text{for any } x \in X \text{ and any } r > 0 \, . \tag{4.31}$$

If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ metric measure space, then by Bishop-Gromov the volume ratio

$$r \mapsto \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} \tag{4.32}$$

is monotone nonincreasing. Hence it makes sense to define

$$\theta_N(x) := \lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} \in (0, +\infty] \, . \tag{4.33}$$

It is elementary to check that

$$\lim_{r \to 0} \frac{v_{K,N}(r)}{\omega_N r^N} = 1 \, , \tag{4.34}$$

for any $K \in \mathbb{R}$ and any $1 \le N < \infty$. Therefore

$$\theta_N(x) = \lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N} \, , \quad \text{for any } x \in X \, . \tag{4.35}$$

End of lecture 4.

Exercise 4.21. Based on the Bishop-Gromov monotonicity, check that the function

$$x \mapsto \theta_N(x) \tag{4.36}$$

is lower semicontinuous on any $\mathrm{RCD}(K, N)$ metric measure space. More in general, prove that if $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x)$ are $\mathrm{RCD}(K, N)$ metric measure spaces converging in the pointed measured Gromov-Hausdorff topology to $(X, \mathsf{d}, \mathfrak{m}, x)$, then for any sequence $z_n \in X_n$ converging to $z \in X$ it holds

$$\theta_N(z) \le \liminf_{n \to \infty} \theta_N(z_n) \, , \tag{4.37}$$

where it is understood that all the densities are relative to the metric measure spaces to which the points belong.

Lemma 4.22. *Let* $(X, \mathsf{d}, \mathscr{H}^N)$ *be an* $\mathrm{RCD}(K, N)$ *metric measure space. Then*

$$\theta_N(x) \le 1 \, , \quad \textit{for any } x \in X \, . \tag{4.38}$$

Proof. It follows from Corollary 4.19 that (4.38) holds for $\mathscr{H}^N$-a.e. $x \in X$ (see also (4.35)). In particular, the set of points for which (4.38) holds is dense in X. The statement follows from the lower semicontinuity of the density (cf. with Exercise 4.21). □

One of the primary applications of the volume cone to metric cone Theorem 4.10 is to the structure of tangent spaces to $\mathrm{RCD}(K, N)$ metric measure spaces $(X, \mathsf{d}, \mathscr{H}^N)$.

Definition 4.23. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space. We say that a pointed metric measure space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$ is a tangent space (or tangent cone) of $(X, \mathsf{d}, \mathfrak{m})$ at some point $x \in X$ if there exists a sequence $r_i \downarrow 0$ such that

$$\left(X, r_i^{-1}\mathsf{d}, \mathfrak{m}/\mathfrak{m}(B_{r_i}(x_i)), x\right) \to (Y, \mathsf{d}_Y, \mathfrak{m}_Y, y) \tag{4.39}$$

as $i \to \infty$ in the pointed measured Gromov-Hausdorff topology.

Remark 4.24. By Gromov's precompactness theorem, the set of tangent spaces at $x \in X$ is always non empty, for any $\mathrm{RCD}(K, N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$. Moreover, by the stability Theorem 2.34 (and the scaling of the lower Ricci bound under scaling of the distance) any tangent space is an $\mathrm{RCD}(0, N)$ metric measure space.

Remark 4.25. For $\mathrm{RCD}(0, N)$ spaces it is possible to introduce in analogous way the notion of tangent cone at infinity (or blow-down) by considering sequences $r_i \uparrow +\infty$.

Exercise 4.26. For smooth N-dimensional Riemannian manifolds all tangent cones are unique and isomorphic to the Euclidean space $\mathbb{R}^N$ with canonical structure.

Remark 4.27. For general $\mathrm{RCD}(K,N)$ metric measure spaces tangent cones do not need to be metric cones. Examples of Ricci limit spaces for which some tangent cone is not a metric cone were constructed by Cheeger-Colding [30, Example 8.95] and Menguy [80].

Exercise 4.28 (Technically demanding). Check that the examples of Ricci limit spaces in [30, 80] are RCD spaces neglecting the existence of approximating sequences of smooth manifolds. See the very recent [43] for the solution.

Remark 4.29. The tangent cone of an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathscr{H}^N)$ at a given point is not unique, in general. The first example of a smooth Riemannian metric on $\mathbb{R}^4$ with positive Ricci curvature, Euclidean volume growth (i.e. $\lim_{r\to\infty}\mathrm{vol}(B_r(p))/\omega_4 r^4>0$), quadratic curvature decay and non unique tangent cones at infinity is due to Perelman (unpublished). Subsequent examples can be found in [30, Example 8.41] for $N=4$ and in the more recent work of Colding-Naber [41] for $N=3$. Notice that tangent cones of $\mathrm{RCD}(K,2)$ metric measure spaces $(X,\mathsf{d},\mathscr{H}^2)$ are unique by the work of Lytchak-Stadler [77].

Theorem 4.30. *Let $(X,\mathsf{d},\mathscr{H}^N)$ be an $\mathrm{RCD}(K,N)$ metric measure space and let $x\in X$. Then any tangent cone at x is a metric measure cone.*

Proof. We claim that for an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathfrak{m})$ and a point $x\in X$ any tangent cone at x is a metric measure cone provided $\theta_N(x)<\infty$. The statement then follows from Lemma 4.22.

Let us consider any tangent cone $(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)$ obtained as a pointed measured Gromov-Hausdorff limit

$$\left(X,r_i^{-1}\mathsf{d},\mathfrak{m}/r_i^N,x\right)\to(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)\,, \tag{4.40}$$

for some sequence $r_i\downarrow 0$. We wish to verify that

$$\frac{\mathfrak{m}_Y(B_r(y))}{\omega_N r^N}=\theta_N(x)\,,\quad \text{for any } 0<r<\infty\,. \tag{4.41}$$

By pointed measured Gromov-Hausdorff convergence

$$\frac{\mathfrak{m}_Y(B_r(y))}{\omega_N r^N}=\lim_{i\to\infty}\frac{\mathfrak{m}(B_{rr_i}(x))}{\omega_N (rr_i)^N}=\theta_N(x)\,, \tag{4.42}$$

for any $r>0$. In particular, (4.41) holds. The conclusion that $(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)$ is a metric measure cone follows from the volume cone implies metric cone Theorem 4.10.

□

Remark 4.31. The role of Bishop-Gromov volume monotonicity in establishing the conicality of tangents for $\mathrm{RCD}(K,N)$ metric measure spaces $(X,\mathsf{d},\mathscr{H}^N)$ is completely analogous to the role of the monotonicity formula for minimal surfaces in Geometric Measure Theory.

Corollary 4.32. *Let (M^N,g) be a smooth Riemannian manifold with nonnegative Ricci curvature. Let d be the induced Riemannian distance. Assume that for some (and hence for every) $p\in M$ it holds*

$$\lim_{R\to\infty}\frac{\mathscr{H}^N(B_R(p))}{\omega_N r^N}>0\,. \tag{4.43}$$

Then any pointed Gromov-Hausdorff limit of a sequence $(M^N,R_i^{-1}\mathsf{d},p)$ for some sequence $R_i\to\infty$ is a metric cone $C(X)$ over a cross section (X,d) which is an $\mathrm{RCD}(N-2,N-1)$ space when endowed with some measure $\mathfrak{m}$.

Definition 4.33. An $\mathrm{RCD}(0,N)$ space $(X,\mathsf{d},\mathscr{H}^N)$ such that

$$\lim_{R\to\infty}\frac{\mathscr{H}^N(B_R(p))}{\omega_N R^N}>0 \tag{4.44}$$

is said to have *Euclidean volume growth.*

Remark 4.34. As we shall see, $\mathfrak{m}=\mathscr{H}^N$ in Corollary 4.32, by the volume convergence Theorem 5.5.

Remark 4.35. Under the assumptions of Corollary 4.32, it is not guaranteed that the cross section of a tangent cone at infinity is isometric to a smooth Riemannian manifold, unless $N=2$.

Exercise 4.36. Find an example of three manifold with nonnegative Ricci curvature and Euclidean volume growth such that the cross section of one of its tangent cones at infinity is not isometric to a smooth 2d Riemannian manifold. Much more delicate constructions can be found in [71] and [41].

Exercise 4.37 (Less elementary). Completely characterize the set of all possible cross sections of tangent cones at infinity of three manifolds with nonnegative Ricci curvature and Euclidean volume growth.

5. Lecture 5

In the last two lectures we shall see how a partial regularity theory for RCD spaces (with the additional $\mathfrak{m}=\mathscr{H}^N$ assumption in force) can be developed.

5.1. Regularity theory.

In analogy with the regularity theory in PDEs and Geometric Measure Theory, it is possible to classify points depending on the behaviour of tangent cones. The first distinction is between *regular* and *singular* points.

Definition 5.1. Given an $\mathrm{RCD}(K,N)$ metric measure space $(X,\mathsf{d},\mathscr{H}^N)$ we say that $x\in X$ is a *regular* point if the tangent cone at x is unique and isomorphic to

$$\left(\mathbb{R}^N,\mathsf{d}_{\mathrm{eucl}},\mathscr{L}^N/\omega_N,0^N\right)\,. \tag{5.1}$$

The set of all regular points $x\in X$ will be denoted by $\mathcal{R}$. The complement of the set of regular points $\mathcal{S}:=X\setminus\mathcal{R}$ is the set of *singular* points.

The first regularity result that we discuss is *rectifiability*, see [78] for a recent survey and [48] for some background about the Euclidean theory. As we shall see, this holds in a strong sense, namely the rectifiable charts can be chosen to have Lipschitz constants arbitrarily close to 1.

Definition 5.2 (Strong rectifiability)**.** Let $(X,\mathsf{d},\mathscr{H}^N)$ be a metric measure space. We say that it is *strongly rectifiable* if for any $\varepsilon>0$ there exists a countable collection of Borel sets $U_k\subset X$ and maps $\varphi_k:U_k\to\mathbb{R}^N$ such that

$$\mathscr{H}^N\left(X\setminus\bigcup_k U_k\right)=0 \tag{5.2}$$

and

$$\varphi_k:U_k\to\varphi_k(U_k)\subset\mathbb{R}^N \tag{5.3}$$

is $(1+\varepsilon)$-biLipschitz, i.e.

$$(1-\varepsilon)\mathsf{d}(x,y)\le|\varphi_k(x)-\varphi_k(y)|\le(1+\varepsilon)\mathsf{d}(x,y)\,, \tag{5.4}$$

for any $x,y\in U_k$ and for any $k\in\mathbb{N}$.

Theorem 5.3 (Rectifiable structure)**.** *Let* $(X, \mathsf{d}, \mathscr{H}^N)$ *be an* $\mathrm{RCD}(K,N)$ *metric measure space for some* $K \in \mathbb{R}$ *and* $1 \le N < \infty$*. Then* $(X, \mathsf{d}, \mathscr{H}^N)$ *is strongly rectifiable. Moreover the set* $\mathcal{R}$ *of regular points has full* $\mathscr{H}^N$*-measure.*

Remark 5.4. More in general, any $\mathrm{RCD}(K,N)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is strongly $(\mathfrak{m}, k)$-rectifiable. This amounts to say that there exists a natural number $1 \le k \le N$, called the *essential dimension* or *rectifiable dimension* of $(X, \mathsf{d}, \mathfrak{m})$, such that the requirements in Definition 5.2 are met with $\mathscr{H}^N$ replaced by $\mathfrak{m}$ and $\mathbb{R}^N$ replaced by $\mathbb{R}^k$. Moreover, $\mathfrak{m} = \theta \mathscr{H}^k$ for some nonnegative density function $\theta \in L^1_{\mathrm{loc}}(\mathscr{H}^k)$.
For Ricci limit spaces this was achieved in [32, 39]. For $\mathrm{RCD}(K,N)$ spaces this is the outcome of [81], [72, 46, 59] and [20]. See also [18] for a subsequent proof of the rectifiability for RCD spaces much more in the spirit of the original one for Ricci limits.

Next we deal with the behaviour of the volume measure $\mathscr{H}^N$ with respect to Gromov-Hausdorff convergence for $\mathrm{RCD}(K,N)$ metric measure spaces $(X, \mathsf{d}, \mathscr{H}^N)$.
The continuity of the volume functional on the space of closed n-manifolds with Ricci curvature bounded below by $-(n-1)$ was conjectured by Cheeger-Anderson and proved by Colding in [38], see also the previous work by Colding [37]. This was later extended by Cheeger-Colding to Ricci limit spaces in [31]. The present formulation for N-dimensional $\mathrm{RCD}(K,N)$ spaces is due to De Philippis-Gigli [45].

Theorem 5.5 (Volume convergence)**.** *Let* $(X_n, \mathsf{d}_n, \mathscr{H}^N, x_n)$ *be a sequence of* $\mathrm{RCD}(K,N)$ *metric measure spaces converging in the pointed Gromov-Hausdorff topology to* (X, d, x) *then one of the following two possibilities occurs:*

i)
$$\limsup_{n\to\infty} \mathscr{H}^N(B_1(x_n)) > 0\,, \tag{5.5}$$
then $(X_n, \mathsf{d}_n, \mathscr{H}^N, x_n)$ *converge in the pointed measured Gromov-Hausdorff topology to* $(X, \mathsf{d}, \mathscr{H}^N, x)$*, which is an* $\mathrm{RCD}(K,N)$ *metric measure space in particular.*

ii)
$$\lim_{n\to\infty} \mathscr{H}^N(B_1(x_n)) = 0\,, \tag{5.6}$$
in which case $\dim_{\mathscr{H}}(X) \le N-1$.

The original formulation of the volume convergence motivated the notion of *non collapsed* Ricci limit space.

Definition 5.6. We say that (X, d, p) is a non collapsed Ricci limit space if there exists a sequence of smooth N-dimensional Riemannian manifolds (M^N_n, g_n) with Ricci curvature bounded from below by $K \in \mathbb{R}$ and points $p_n \in M^N_n$ such that
$$\mathscr{H}^N(B_1(p_n)) > v > 0\,, \tag{5.7}$$
for any $n \in \mathbb{N}$ and $(M^N_n, \mathsf{d}_n, p_n) \to (X, \mathsf{d}, p)$ in the pointed Gromov-Hausdorff sense.

Remark 5.7. By the volume convergence Theorem 5.5, under the assumptions above the sequence $(M^N_n, \mathsf{d}_n, \mathscr{H}^N, p_n)$ converges in the pointed measured Gromov-Hausdorff sense to $(X, \mathsf{d}, \mathscr{H}^N, p)$. In particular, any non collapsed Ricci limit space is an N-dimensional $\mathrm{RCD}(K,N)$ metric measure space.

Topological stability theorems for manifolds subject to Ricci curvature bounds were originally considered by Anderson [11], who proved that for a ball $B_1(p)$ in an N-dimensional Riemannian manifold with nonnegative Ricci curvature verifying an additional upper Ricci curvature bound $\mathrm{Ric} \le C$ and with almost maximal volume (i.e. $\mathrm{vol}(B_1(p) > \omega_N - \varepsilon_N)$ the smaller ball $B_{1/2}(p)$ is diffeomorphic to the Euclidean ball with metric close to the Euclidean one in $C^{1,\alpha}$.

Remark 5.8. To some extent, [11, Theorem 3.2] is an analogue of Allard's regularity theorem for Euclidean submanifolds with bounded mean curvature in Geometric Measure Theory.

Later in [86] Perelman removed the upper Ricci curvature bound from the assumptions in the case of a positive Ricci curvature lower bound (so the model is the sphere) and proved contractibility in the case of nonnegative Ricci. With the volume convergence Theorem 5.5, almost maximality of the volume can be replaced by closeness to the model ball in the Gromov-Hausdorff sense, see [37]. This assumption turns to be satisfied at most locations and scales, in view of Theorem 5.3.

Remark 5.9. In [86] (under a positive lower Ricci bound) the homeomorphism is obtained in indirect way. Indeed, it is proved that the homotopy groups $\pi_i(M^N)$ vanish for any $i < N$. Then the homeomorphism with the sphere follows from the work of Hamilton [66] in dimension $N = 3$, of Smale [92] and Freedman [50] in dimension $N \geq 4$.

Later in [30], Cheeger-Colding formulated an intrinsic version of Reifenberg's theorem [89] for metric spaces. With the help of the almost Euclidean volume rigidity and of Theorem 5.3, this allows to improve up to diffeomorphic stability the topological stability theorems discussed above. Moreover, it is possible to prove a topological manifold structure theorem for non collapsed Ricci limit spaces, away from a set of Hausdorff codimension two, see [30, Section 5].

Theorem 5.10 (Manifold structure)**.** *Let $(X, \mathsf{d}, \mathscr{H}^N)$ be an $\mathrm{RCD}(K,N)$ metric measure space for some $K \in \mathbb{R}$ and $1 \leq N < \infty$. Then for any $\alpha \in (0,1)$ there exists an open set $U_\alpha \subset X$ with the following properties:*

- i) *$\mathcal{R} \subset U_\alpha \subset X$;*
- ii) *U_α is a C^α-manifold, i.e. it is homeomorphic to a smooth differentiable manifold with charts $\varphi^k_\alpha : U^k_\alpha \to \mathbb{R}^N$ that are C^α-biHölder homeomorphisms with their images.*

The distinction between regular and singular points can be refined further through the so called stratification of singularities.

Definition 5.11. For any $0 \leq k \leq N-1$ we introduce the k-th singular stratum $\mathcal{S}^k \subset \mathcal{S}$ by

$$\mathcal{S}^k := \left\{ x \in X \, : \, \text{no tangent cone at } x \text{ splits a factor } \mathbb{R}^{k+1} \right\} . \tag{5.8}$$

It follows by the very definition of the k-th singular strata that there is a filtration of the singular set

$$\mathcal{S}^0 \subseteq \mathcal{S}^1 \subseteq \cdots \subseteq \mathcal{S}^{N-1} \subseteq X \, . \tag{5.9}$$

Moreover, $\mathcal{S}^{N-1} = \mathcal{S}$, thanks to the following Euclidean volume rigidity and the volume convergence Theorem 5.5.

Theorem 5.12. *Let $(X, \mathsf{d}, \mathscr{H}^N)$ be an $\mathrm{RCD}(0,N)$ metric measure space such that*

$$\frac{\mathscr{H}^N(B_r(p))}{\omega_N r^N} = 1 \, , \quad \text{for any } 0 < r < \infty \, , \tag{5.10}$$

for some $p \in X$. Then $(X, \mathsf{d}, \mathscr{H}^N)$ is isomorphic to the N-dimensional Euclidean space.

Proof. We prove the Euclidean volume rigidity in two steps. In the first one we prove that (5.10) forces $(X, \mathsf{d}, \mathscr{H}^N)$ to be a cone with respect to any base point. In the second step we shall see that this is enough to obtain the Euclidean rigidity.

It is an easy consequence of Bishop-Gromov that the asymptotic volume ratio

$$\lim_{R \to \infty} \frac{\mathscr{H}^N(B_R(p))}{\omega_N R^N} \tag{5.11}$$

is independent of the chosen base point $p \in X$ for an $\mathrm{RCD}(0, N)$ metric measure space $(X, \mathsf{d}, \mathscr{H}^N)$. By (5.10),

$$\lim_{R\to\infty} \frac{\mathscr{H}^N(B_R(p))}{\omega_N R^N} = 1\,, \tag{5.12}$$

for any $p \in X$. By Bishop-Gromov monotonicity and Lemma 4.22,

$$\frac{\mathscr{H}^N(B_R(p))}{\omega_N R^N} = 1\,, \quad \text{for any } p \in X \text{ and any } 0 < R < \infty\,. \tag{5.13}$$

The volume cone implies metric cone Theorem 4.10 implies then that $(X, \mathsf{d}, \mathscr{H}^N)$ is a metric measure cone with respect to any base point $p \in X$.

The Euclidean rigidity now follows by iterating finitely many times the following simple observation. If $(X, \mathsf{d}, \mathscr{H}^N)$ is a cone with respect to points $x_1, x_2 \in X$ with $x_1 \neq x_2$, then x_1 and x_2 are intermediate points of a line. In particular by the splitting Theorem 3.10 $(X, \mathsf{d}, \mathscr{H}^N)$ splits a line isometrically.
In order to check the claim we notice that by conicality with respect to x_1, x_2 belongs to a ray with end point x_1. In particular, any tangent cone of $(X, \mathsf{d}, \mathscr{H}^N)$ at x_2 splits a line isometrically, by the splitting Theorem 3.10. Since $(X, \mathsf{d}, \mathscr{H}^N)$ is a metric measure cone with respect to the base point x_2, any tangent cone at x_2 is isomorphic to $(X, \mathsf{d}, \mathscr{H}^N, x_2)$. Hence $(X, \mathsf{d}, \mathscr{H}^N)$ splits a line. □

Exercise 5.13. Find a more elementary argument for the second part of the proof of Theorem 5.12.

Remark 5.14. There is a local version of the above Euclidean volume rigidity, where the assumption is that

$$\frac{\mathscr{H}^N(B_R(p))}{\omega_N R^N} = 1\,, \quad \text{for some } p \in X \text{ and some } R > 0 \tag{5.14}$$

and the conclusion is that the ball $B_{R/2}(p)$ is isometric to the Euclidean ball. Notice that it is necessary to decrease the radius to find the isometry, as elementary examples show.

Based on Federer's dimension reduction, the splitting theorem, the volume convergence and the volume cone implies metric cone theorem, one can prove Hausdorff dimension bounds for the singular strata of $\mathrm{RCD}(K, N)$ metric measure spaces $(X, \mathsf{d}, \mathscr{H}^N)$. The analogous statement for non collapsed Ricci limit spaces was proved by Cheeger-Colding in [30]. The present formulation is due to De Philippis-Gigli [45].

Theorem 5.15 (Stratification of singularities)**.** *Let $(X, \mathsf{d}, \mathscr{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space. Then for any $0 \le k \le N-1$ it holds*

$$\dim_{\mathscr{H}} \left(\mathcal{S}^k\right) \le k\,. \tag{5.15}$$

5.2. **Examples.** We discuss a series of examples illustrating to what extent the statements discussed above are sharp and some the most recent developments in related directions.

Example 5.16. It is possible to find a three dimensional, non collapsed, pointed Ricci limit space $(\mathbb{R}^3, \mathsf{d}, p)$ with the following property: any point $q \in X$ is a regular point according to Definition 5.1. However, there is no neighbourhood U of $p \in X$ such that the restriction of distance d to $U' \subset U$ is induced by a Hölder Riemannian metric on U. See [40] for the construction.

Remark 5.17. The strong rectifiability of $\mathrm{RCD}(K, N)$ metric measure spaces in the sense of Definition 5.2 is tightly linked with their infinitesimal Hilbertianity.

Exercise 5.18. Let $\|\cdot\|$ be any norm on $\mathbb{R}^N$ with induced distance d. Check that the metric measure space $(\mathbb{R}^N, \mathsf{d}, \mathscr{L}^N)$ is strongly rectifiable if and only if $\|\cdot\|$ is induced by a scalar product.

Remark 5.19. The metric measure space $(\mathbb{R}^N, \mathsf{d}, \mathscr{L}^N)$ discussed above is $\mathrm{CD}(0,N)$ for any choice of the norm $\|\cdot\|$, see [95, Conclusions and open problems]. It is $\mathrm{RCD}(0,N)$ if and only if the norm $\|\cdot\|$ is induced by a scalar product.

Example 5.20 (Goose bumps). This example is borrowed from [37]. It is possible to construct metrics on the two-sphere $(\mathbb{S}^2, g_n)$ by attaching n^2 spheres with volume $1/n^2$ through very small necks. This can be done so that

$$\mathscr{H}^2(\mathbb{S}^2, g_n) = \mathscr{H}^2(\mathbb{S}^2, g_{\mathrm{can}}) + 1\,, \tag{5.16}$$

and

$$(\mathbb{S}^2, g_n) \to\to (\mathbb{S}^2, g_{\mathrm{can}}) \tag{5.17}$$

in the Gromov-Hausdorff sense.

Remark 5.21. The possibility to improve the biHölder regularity of the charts in Theorem 5.10 to biLipschitz has remained an open question since [30].

Notice that there is a subtlety in Definition 5.11. Namely the k-th stratum $\mathcal{S}^k \setminus \mathcal{S}^{k-1}$ is defined to be the set of those points where no tangent cone splits $\mathbb{R}^{k+1}$ but at least one tangent splits a factor $\mathbb{R}^k$. This means that the maximal Euclidean factor of the tangent cones on $\mathcal{S}^k \setminus \mathcal{S}^{k-1}$ is not a priori fixed and indeed there are examples where it is not.

Example 5.22. For any $N \geq 3$ and any $0 \leq k \leq N-2$ there exists an example of N-dimensional non collapsed Ricci limit space (X, d, x) such that at the point x there are different tangent cones whose maximal Euclidean factor is exactly $\mathbb{R}^k$.

Remark 5.23. For $k = N$ and $k = N-1$, there is much more rigidity. In the top dimensional case Theorem 5.12 and the volume convergence say that a point such that one tangent cone is $\mathbb{R}^N$ is actually regular. In the codimension one case:

- in [30] it was proved that for non collapsed N-dimensional Ricci limit spaces (where it is assumed that the smooth manifolds in the sequence have no boundary) $\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2} = \emptyset$. In particular the improved Hausdorff dimension estimate
$$\dim_{\mathscr{H}}(\mathcal{S}) \leq N-2 \tag{5.18}$$
for the singular set holds in this case;
- more recently in [17] it has been shown that for any $\mathrm{RCD}(K,N)$ metric measure space $(X, \mathsf{d}, \mathscr{H}^N)$ and any point $x \in \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}$ all the tangent cones at x are Euclidean half-spaces.

The structure theory for the singular strata $\mathcal{S}^k$ for non collapsed Ricci limit spaces has been dramatically improved in a recent paper by Cheeger-Jiang-Naber [34] where the following has been proved, among the others.

Theorem 5.24. *Let (X, d) be a Gromov-Hausdorff limit of a sequence of manifolds (M_n^N, g_n) with $\mathrm{Ric}_n \geq -(N-1)$ and*

$$\mathrm{vol}(B_1(p_n)) > v > 0\,. \tag{5.19}$$

Then the k-th singular stratum $\mathcal{S}^k(X)$ is k-rectifiable and for $\mathscr{H}^k$-a.e. $x \in \mathcal{S}^k(X)$ any tangent cone at x splits a factor $\mathbb{R}^k$ isometrically.

The generalization of Theorem 5.24 to $\mathrm{RCD}(K,N)$ spaces $(X, \mathsf{d}, \mathscr{H}^N)$ is currently an open problem.

5.3. **Stratification of the singular set.** Two direct consequences of the definitions valid for general metric spaces (X, d) are:

$$\mathscr{H}^k(E) = 0 \Leftrightarrow \mathscr{H}^k_\infty(E) = 0\,, \tag{5.20}$$

for any $E \subset X$ and any $k \geq 0$ and

$$\mathsf{d}_H(E_n, E) \to 0 \Rightarrow \mathscr{H}^k_\infty(E) \geq \limsup_{n\to\infty} \mathscr{H}^k_\infty(E_n)\,, \tag{5.21}$$

whenever $E \subset X$ is compact. Moreover, we will employ the following classical result relating Hausdorff measures and densities with respect to the Hausdorff pre-measure, see for instance [48, Theorem 2.10.17].

Lemma 5.25. *Let (X, d) be a metric space and $E \subset X$ be a Borel set. Let $k \geq 0$. Then for $\mathscr{H}^k$-a.e. $x \in E$ it holds*

$$\limsup_{r\to 0} \frac{\mathscr{H}^k_\infty(E \cap B_r(x))}{r^k} \geq \frac{\omega_k}{2^k}\,. \tag{5.22}$$

The cone splitting principle plays a key role in the argument. This was implicitly formulated in the proof of Theorem 5.12, but let us state it precisely and in higher generality, cf. with [35]. The proof is elementary and we omit it.

Lemma 5.26 (Cone splitting)**.** *Let $(Z, \mathsf{d}_Z), (\bar{Z}, \mathsf{d}_{\bar{Z}})$ be metric spaces and let $k \in \mathbb{N}$. Assume that there is an isometry $I : C(Z) \times \mathbb{R}^k \to C(\bar{Z})$, where $C(Z)$ and $C(\bar{Z})$ are metric cones with respective vertexes z and $\bar{z}$ respectively, and*

$$\bar{z} \notin I\left(\mathbb{R}^k \times C(Z)\right)\,. \tag{5.23}$$

Then there exists a metric space (W, d_W) such that $\mathbb{R}^k \times C(Z)$ is isometric to $\mathbb{R}^{k+1} \times C(W)$.

Exercise 5.27. It is not a priori true that limits of regular points are regular points and limits of singular points are singular points. Find counterexamples.

Proof of Theorem 5.15. The proof is based on a *dimension reduction* type argument originating in Federer's proof of the analogous Hausdorff dimension estimate for area minimizing currents in codimension one, see [49].

Let us assume by contradiction that (5.15) fails for some $\mathrm{RCD}(K, N)$ metric measure space $(X, \mathsf{d}, \mathscr{H}^N)$ and some $0 \leq k \leq N - 1$. By the very definition of Hausdorff dimension there exists $k' > k$ such that

$$\mathscr{H}^{k'}(\mathcal{S}^k(X)) > 0\,. \tag{5.24}$$

We can further refine the stratification of the singular set by introducing the effective singular stratum $\mathcal{S}^k_\varepsilon(X)$ as the collection of those points $x \in X$ such that for any radius $0 < r < \varepsilon$ the ball $B_r(x)$ is εr-away in Gromov-Hausdorff distance from any ball $B_r(0, x') \subset \mathbb{R}^{k+1} \times Z$ for some metric space (Z, d_Z). It is a simple exercise to check that $\mathcal{S}^k_\varepsilon$ is closed for any $\varepsilon > 0$ and

$$\mathcal{S}^k = \bigcup_{\varepsilon > 0} \mathcal{S}^k_\varepsilon\,. \tag{5.25}$$

In particular, if (5.24) holds, then there exists $\bar{\varepsilon} > 0$ such that

$$\mathscr{H}^{k'}\left(\mathcal{S}^k_{\bar{\varepsilon}}\right) > 0\,. \tag{5.26}$$

By Lemma 5.25, there exist $\bar{x} \in \mathcal{S}^k_{\bar{\varepsilon}}$ and a sequence $r_i \downarrow 0$ such that

$$\lim_{i\to\infty} \frac{\mathscr{H}^{k'}_\infty(\mathcal{S}^k_{\bar{\varepsilon}} \cap B_{r_i}(\bar{x}))}{r^{k'}} \geq \frac{\omega_{k'}}{2^{k'}}\,. \tag{5.27}$$

We rescale the metric measure space $(X, \mathsf{d}, \mathscr{H}^N)$ based at $\bar{x}$ along the sequence $r_i \downarrow 0$. Up to the extraction of a subsequence that we do not relabel,

$$(X, r_i^{-1}\mathsf{d}, \mathscr{H}^N/r_i^N, x) \to (Y, \mathsf{d}_Y, \mathscr{H}^N, y)\,, \tag{5.28}$$

in the pointed measured Gromov-Hausdorff sense, by Theorem 5.5. By Blaschke's compactness theorem for metric spaces, see [21, Theorem 3.7.8], we can assume that, up to extraction of a subsequence that we do not relabel, the compact sets

$$\mathcal{S}^k_{\bar{\varepsilon}} \cap B_1^{\mathsf{d}_i}(\bar{x}) \tag{5.29}$$

converge to a compact set $A \subset B_1^{\mathsf{d}_Y}(y)$, with respect to the Hausdorff distance. Here we assume that the convergence in (5.28) is realized by isometric embeddings into a common metric space (Z, d_Z) and we denote $\mathsf{d}_i := r_i^{-1}\mathsf{d}$ and $X_i := (X, \mathsf{d}_i)$ for brevity.
It follows from the very definition of the effective singular stratum that $A \subset \mathcal{S}^k_{\bar{\varepsilon}}(Y)$. Hence by (5.21)

$$\mathscr{H}^{k'}_{\infty}(\mathcal{S}^k_{\bar{\varepsilon}}(Y)) \geq \mathscr{H}^{k'}_{\infty}(A) \geq \limsup_{i\to\infty} \mathscr{H}^{k'}_{\infty}(\mathcal{S}^k_{\bar{\varepsilon}}(X_i) \cap B_1^{\mathsf{d}_i}(\bar{x})) \tag{5.30}$$

$$= \limsup_{i\to\infty} \frac{\mathscr{H}^{k'}_{\infty}(\mathcal{S}^k_{\bar{\varepsilon}}(X) \cap B_{r_i}(\bar{x}))}{r_i^{k'}} > 0\,. \tag{5.31}$$

By Theorem 4.30, $(Y, \mathsf{d}_Y, \mathscr{H}^N, y)$ is a metric measure cone. By (5.20),

$$\mathscr{H}^{k'}\left(\mathcal{S}^k_{\bar{\varepsilon}}(Y) \setminus \{y\}\right) > 0\,. \tag{5.32}$$

Then we can iterate the argument blowing-up $(Y, \mathsf{d}_Y, \mathscr{H}^N, y)$ at some $z \neq y$, $z \in \mathcal{S}^k_{\varepsilon}(Y)$. We obtain a tangent cone $(Y_1, \mathsf{d}_{Y_1}, \mathscr{H}^N, y_1)$ with the property that

$$\mathscr{H}^{k'}\left(\mathcal{S}^k_{\bar{\varepsilon}}(Y_1)\right) > 0 \tag{5.33}$$

and Y_1 splits a line isometrically, as it is the blow-up of a cone based at a point different from the tip.
If $k = 0$, this is a contradiction with the definition of $\mathcal{S}^k_{\varepsilon}$, as we found a tangent cone at z splitting a line. Otherwise, $k \geq 1$ and $k' > 1$. Then we can write

$$Y_1 = \mathbb{R} \times Y_2 \tag{5.34}$$

where $(Y_2, \mathsf{d}_2, \mathscr{H}^{N-1}, y_2)$ is an $\mathrm{RCD}(0, N-1)$ metric measure space and notice that

$$(r, x) \in \mathcal{S}^k(Y_1) \;\Leftrightarrow\; x_1 \in \mathcal{S}^{k-1}(Y_2)\,. \tag{5.35}$$

Moreover, it is a classical property of Hausdorff measures that

$$\mathscr{H}^{k-1}(Z) > 0 \Leftrightarrow \mathscr{H}^k(\mathbb{R} \times Z) > 0\,. \tag{5.36}$$

The combination of (5.35) and (5.36) with (5.33) and (5.34) shows that

$$\dim_{\mathscr{H}}\left(\mathcal{S}^{k-1}(Y_2)\right) > k - 1\,. \tag{5.37}$$

The argument can be iterated finitely many times to reach a contradiction.

□

6. Lecture 6

In this lecture we discuss some of the tools that are involved in the proofs of the rectifiable structure Theorem 5.3, the topological structure Theorem 5.10 and the volume convergence Theorem 5.5 for $\mathrm{RCD}(K, N)$ metric measure spaces $(X, \mathsf{d}, \mathscr{H}^N)$.
Often the building blocks for the regularity theory are so-called ε-regularity theorems, cf. with the discussion in [27]. These are local stability theorems asserting that, under suitable assumptions, once some space looks close to a model in some weak sense, then on a subregion with smaller but *definite* size, the space is going to look close to the model in a

much stronger sense. A prototype is the ε-regularity theorem for manifolds with bounded Ricci curvature from [11].

In the present setting, one considers a ball $B_4(p) \subset X$ (there is no loss of generality in doing so, as all the statements are scale invariant) where $(X, \mathsf{d}, \mathscr{H}^N)$ is an $\mathrm{RCD}(-\delta, N)$ metric measure space and such that

$$\mathsf{d}_{GH}\left(B_4(p), B_4(0)\right) \le \delta \ll 1\,, \tag{6.1}$$

with $B_4(0) \subset \mathbb{R}^N$. Under this assumption we shall see that the following hold:

i)
$$\left|\mathscr{H}^N(B_1(p)) - \omega_N\right| \le \varepsilon(\delta, N)\,, \tag{6.2}$$
with $\varepsilon(\delta, N) \to 0$ as $\delta \to 0$;

ii) there exists a map $u : B_2(p) \to B_3(0) \subset \mathbb{R}^N$ with open image and which is $(1 + C(N)\varepsilon(\delta, N))$-Lipschitz and a biHölder homeomorphism with its image;

iii) there exists a set $E \subset B_2(p)$ with
$$\mathscr{H}^N(B_2(p) \setminus E) \le \varepsilon(\delta, N)\,, \tag{6.3}$$
such that the map u in (ii) is $(1 + \varepsilon(\delta, N))$-biLipschitz when restricted to E.

Our focus will be on (i), (ii), (iii). The reduction of the structure theorems and of the volume convergence to their local versions (i), (ii) and (iii) is based on a series of covering arguments that ultimately boil down to the doubling property of the reference measure $\mathscr{H}^N$ and to the iterative application of the splitting theorem.

We rely on a variant for doubling metric measure spaces of the classical Hardy-Littlewood theorem on the boundedness of the maximal function between L^p spaces, see for instance [10, Chapter 5] for a proof. We introduce the notion of *maximal function.*

Definition 6.1. Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Let $f \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$ be nonnegative. Then the *maximal function* $\mathrm{M}f : X \to [0, \infty]$ is defined by

$$\mathrm{M}f(x) := \sup_{r>0} \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m} = \sup_{r>0} \frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m}\,. \tag{6.4}$$

Analogously, we shall denote $\mathrm{M}_R f : X \to [0, \infty]$ the *restricted* maximal function defined by

$$\mathrm{M}_R f(x) := \sup_{0<r<R} \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m}\,. \tag{6.5}$$

Theorem 6.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally uniformly doubling metric measure space, i.e. we assume that there exists $C : [0, \infty) \to [0, \infty)$ such that*

$$\mathfrak{m}(B_{2r}(x)) \le C(R)\mathfrak{m}(B_r(x))\,, \quad \text{for any } x \in X \text{ and any } 0 < r < R\,. \tag{6.6}$$

Then the following hold:

i) *for any $1 < p < \infty$ there exists a function $C'_p : [0, \infty) \to [0, \infty)$ such that*
$$\left\|\mathrm{M}_r f\right\|_{L^p(B_r(x))} \le C'_p(R) \left\|f\right\|_{L^p(B_{2r}(x))}\,, \quad \text{for any } x \in X \text{ and any } 0 < r < R\,; \tag{6.7}$$

ii) *Lebesgue differentiation theorem holds, namely for any function $f \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$ it holds*
$$\lim_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \, \mathrm{d}\mathfrak{m}(y) = 0\,, \tag{6.8}$$
for $\mathfrak{m}$-a.e. $x \in X$.

Remark 6.3. The uniform local doubling assumption, although very general, is strong enough to make the classical Euclidean proofs of the maximal function theorem and of the Lebesgue differentiation theorem carry over to this setting.

Notice that $\mathrm{RCD}(K,N)$ metric measure spaces are locally uniformly doubling thanks to Bishop-Gromov Theorem 2.8. The combination of Lebsegue's differentiation theorem with the Euclidean volume rigidity Theorem 5.12 has the following interesting consequence.

Proposition 6.4. *Let $(X,\mathsf{d},\mathscr{H}^N)$ be an $\mathrm{RCD}(K,N)$ metric measure space. Then $\mathscr{H}^N$-a.e. $x\in X$ is a regular point.*

Proof. We outline the argument borrowed from [45, Proposition 2.20]. The conclusion follows from the observation that at any approximate continuity point x of the density function Θ_N the assumptions of Theorem 5.12 are met for any tangent cone at x.

The first step is to employ Lemma 4.22 in combination with Theorem 6.2 (ii) to check that $\mathscr{H}^N$-a.e. $x\in X$ is an approximate continuity point of Θ_N, i.e.

$$\lim_{r\to 0}\frac{\mathscr{H}^N\left(\{y\in B_r(x)\,:\,|\Theta_N(x)-\Theta_N(y)|>\varepsilon\}\right)}{\mathscr{H}^N(B_r(x))}=0\,, \tag{6.9}$$

for any $\varepsilon>0$.

Our next goal is to prove that at any point x as in (6.9), any tangent cone is a metric measure cone with respect to any base point. We let $(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)$ be any tangent cone at x obtained by scaling along the sequence $r_n\downarrow 0$. We claim that any point $\tilde{y}\in Y$ can be approximated in Gromov-Hausdorff sense by a sequence $y_n\in X_n:=(X,r_n^{-1}\mathsf{d})$ with the additional property that

$$\lim_{n\to\infty}\Theta_N^{X_n}(y_n)=\Theta_N^X(x)\,. \tag{6.10}$$

The claim can be checked relying on (6.9).

By (6.10) and Bishop-Gromov monotonicity,

$$\frac{\mathfrak{m}_Y(B_\rho(\bar{y}))}{\omega_N\rho^N}\le\Theta_N^X(x)\,,\quad \text{for any } y\in Y \text{ and any } 0<\rho<\infty\,. \tag{6.11}$$

On the other hand, the converse inequality follows from Theorem 4.30. Namely, $(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)$ is a metric measure cone with tip y and

$$\frac{\mathfrak{m}_\infty(B_R(y))}{\omega_N R^N}=\Theta_N^X(x)\,,\quad \text{for any } 0<R<\infty\,. \tag{6.12}$$

By Bishop-Gromov

$$\lim_{R\to\infty}\frac{\mathfrak{m}_Y(B_R(\bar{y}))}{\omega_N R^N}=\lim_{R\to\infty}\frac{\mathfrak{m}_\infty(B_R(y))}{\omega_N R^N}\,,\quad \text{for any } \bar{y}\in Y\,. \tag{6.13}$$

Hence by (6.12) and Bishop-Gromov again

$$\frac{\mathfrak{m}_Y(B_R(\bar{y}))}{\omega_N R^N}\ge\Theta_N^X(x)\,,\quad \text{for any } R>0 \text{ and any } \bar{y}\in Y\,. \tag{6.14}$$

Combined with (6.11) and the volume cone implies metric cone Theorem 4.10 this shows that $(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)$ is a metric measure cone with respect to any base point. By Theorem 5.12 it is isomorphic to the Euclidean $\mathbb{R}^N$. □

Remark 6.5. The primary outcome of Proposition 6.4 above is that the assumption in (6.1) is in force at most locations and at any sufficiently small scales, after rescaling.

6.1. **Harmonic almost splitting maps.** Our strong goal is to prove that under the assumption (6.1) there exists a map $u:B_2(p)\to\mathbb{R}^N$ which is a biHölder homeomorphism with its image, biLipschitz on a set of large measure and to use it to control the volume gap as in (6.1). The constructed u will have harmonic components.

Remark 6.6. This is again in analogy with the lower bounds on the harmonic radius for balls with almost maximal volume on manifolds with bounded Ricci curvature, see [11].

Theorem 6.7. *Let $N \geq 1$ be fixed. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N) > 0$ such that the following holds. If $(X, \mathsf{d}, \mathscr{H}^N)$ is an $\mathrm{RCD}(-\delta, N)$ metric measure space $p \in X$ and*

$$\mathsf{d}_{mGH}\left(B_4(p), B_4(0^N)\right) \leq \delta \,, \tag{6.15}$$

then there exists a function $u : B_3(p) \to \mathbb{R}^N$ with the following properties:

i) *$u_i : B_3(p) \to \mathbb{R}$ is harmonic for any $i = 1, \dots, N$;*

ii)
$$\sum_{i,j} \fint_{B_2(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \, \mathrm{d}\mathscr{H}^N \leq \varepsilon \,; \tag{6.16}$$

iii)
$$\sup_{B_1(p)} |\nabla u| \leq 1 + C_N \varepsilon \,; \tag{6.17}$$

iv)
$$\sum_i \fint_{B_1(p)} |\mathrm{Hess}\, u_i|^2 \, \mathrm{d}\mathscr{H}^N \leq C_N \varepsilon \,. \tag{6.18}$$

We illustrate the main steps in the proof of Theorem 6.7.
For the existence of the harmonic map satisfying (6.16) there are at least three possibilities. The original argument in [29] uses the harmonic replacement of distance functions in combination with the Abresch-Gromoll inequality from [1] to show that there is the sought integral gradient control. See also the more recent [97].
It is also possible to prove (ii) arguing by contradiction, building on top of the convergence and stability of harmonic functions developed in [8]. The idea is that the components of u approximate the canonical coordinates on $\mathbb{R}^N$ for which (6.16) holds with $\varepsilon = 0$. This is the approach pursued for instance in [18, 19].
The third path is recently adopted by Cheeger-Jiang-Naber in [34]. Splitting maps are built as combinations of solutions of $\Delta v = 2N$ that approximate distance functions squared from points. Notice that in the rigid Euclidean case the coordinates can be obtained in this way, as $\Delta \mathsf{d}_q^2 = 2N$ for any point q.

The elliptic regularity for spaces with lower Ricci curvature bounds shows that the components of u are C_N-Lipschitz, which would be enough for several applications. The argument for the improved Lipschitz bound in (iii) is due to Cheeger-Naber in [36]. It is based on Bochner's inequality

$$\Delta \frac{1}{2} |\nabla u|^2 \geq -\delta \, |\nabla u|^2 \,, \tag{6.19}$$

combined with heat flow techniques.

In all this, a fundamental tool is the existence of good cut-off functions with controlled gradient and Laplacian on spaces with lower Ricci bounds, originally due to Cheeger-Colding [30, Lemma 4.65]. We refer to [81, Lemma 3.1] for the present setting (with a different proof).

Proposition 6.8. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space. For any $R > 0$ there exists a constant $C(K, N, R) > 0$ such that for any $x \in X$ and for any $0 < r < R$ there exists a cut-off function $\varphi_{x,r}$ such that*

$$0 \leq \varphi_{x,r} \leq 1 \,, \quad \varphi_{x,r} \equiv 0 \,, \quad \text{on } X \setminus B_{2r}(x) \,, \quad \varphi_{x,r} \equiv 1 \,, \quad \text{on } B_r(x) \tag{6.20}$$

and

$$r \sup |\nabla \varphi_{x,r}| + r^2 \sup |\Delta \varphi_{x,r}| \leq C(K, N, R) \,. \tag{6.21}$$

The Hessian estimate (6.18) can be obtained from the L^2-gradient estimate by integrating the Bochner inequality against a good cut-off function as in Proposition 6.8. In detail, we start from

$$\Delta\frac{1}{2}\left|\nabla u\right|^2 \geq \left\|\mathrm{Hess}\, u\right\|^2_{\mathrm{HS}} - \delta\left|\nabla u\right|^2 . \tag{6.22}$$

Then we choose a regular cut-off function φ from Proposition 6.8 with respect to $B_1(p) \subset B_2(p) \subset X$, multiply both sides of (6.22) against φ and integrate by parts. We get

$$\begin{aligned}\frac{1}{2}\int_X \Delta\varphi\left|\nabla u\right|^2 \mathrm{d}\mathscr{H}^N &\geq \int_X \varphi\left\|\mathrm{Hess}\, u\right\|^2_{\mathrm{HS}} \mathrm{d}\mathscr{H}^N - C_N\mathscr{H}^N(B_2(p))\\ &\geq \int_{B_1(p)} \left\|\mathrm{Hess}\, u\right\|^2_{\mathrm{HS}} - C_N\delta\mathscr{H}^N(B_2(p)) .\end{aligned}$$

Equivalently

$$\frac{1}{2}\int_X \Delta\varphi\left[\left|\nabla u\right|^2 - 1\right] \mathrm{d}\mathscr{H}^N \geq \int_{B_1(p)} \left\|\mathrm{Hess}\, u\right\|^2_{\mathrm{HS}} - C_N\delta\mathscr{H}^N(B_2(p)) . \tag{6.23}$$

Dividing both sides by $\mathscr{H}^N(B_2(p))$, taking into account Bishop-Gromov monotonicity and the integral gradient bound (6.16) we obtain

$$\fint_{B_1(p)} \left\|\mathrm{Hess}\, u\right\|^2_{\mathrm{HS}} \mathrm{d}\mathscr{H}^N \leq C(N)\fint_{B_2(p)} \left|\left|\nabla u\right|^2 - 1\right| \mathrm{d}\mathscr{H}^N + C(N)\delta \leq C_N\varepsilon , \tag{6.24}$$

if δ is small enough.

6.2. **Rectifiable structure.** The proof of the rectifiable structure Theorem 5.3 is based on the combination of Proposition 6.4 with the following ε-regularity proposition, to the effect that on almost Euclidean balls, the harmonic splitting coordinates obtained via Theorem 6.7 are biLipschitz on a set of almost full measure.

Proposition 6.9. *For any $\varepsilon' > 0$ there exists $\delta(\varepsilon', N) > 0$ such that, if $\delta \leq \delta(\varepsilon, N)$ in Theorem 6.7, then there exists a set $E_{\varepsilon'} \subset B_2(p)$ such that*

i)
$$\mathscr{H}^N(B_2(p) \setminus E_{\varepsilon'}) \leq \varepsilon' ; \tag{6.25}$$

ii) *if $u : B_3(p) \to \mathbb{R}^N$ is the harmonic map obtained via Theorem 6.7, then*
$$(1 - \varepsilon')\mathsf{d}(x, y) \leq |u(x) - u(y)| \leq (1 + \varepsilon')\mathsf{d}(x, y) , \quad \textit{for any } x, y \in E_{\varepsilon'} . \tag{6.26}$$

Sketch of proof. We consider the almost splitting coordinates built in Theorem 6.7 for ε small to be fixed later. An application of the maximal function estimate from Theorem 6.2 shows that the set $E_{\varepsilon''} \subset B_1(p)$ defined to be the set of all those points for which

$$\sup_{0<r<2} \fint \sum_{i,j} \left|\nabla u_i \cdot \nabla u_j - \delta_{ij}\right| \mathrm{d}\mathscr{H}^N \leq \varepsilon'' \tag{6.27}$$

has almost full measure if ε is small enough depending on ε''. If ε'' is small enough depending on ε', then we can apply Proposition 6.10 below to show that for any $x \in E_{\varepsilon''}$ and for any $0 < r < 1$, the map u restricted to $B_r(x)$ is an $\varepsilon' r$-GH isometry. The conclusion then follows from Lemma 6.11 below. □

Proposition 6.10. *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N) > 0$ such that the following holds. If $(X, \mathsf{d}, \mathscr{H}^N)$ is an $\mathrm{RCD}(-\delta, N)$ metric measure space, $p \in X$ and $u : B_3(p) \to \mathbb{R}^N$ is a harmonic map such that $u(p) = 0^N$ and*

$$\sum_{i,j} \fint_{B_2(p)} \left|\nabla u_i \cdot \nabla u_j - \delta_{ij}\right| \mathrm{d}\mathscr{H}^N \leq \delta , \tag{6.28}$$

then u is an ε-GH isometry on $B_1(p)$, namely

$$\left|\left|u(x) - u(y)\right| - \mathsf{d}(x, y)\right| \leq \varepsilon , \quad \textit{for any } x, y \in B_1(p) \tag{6.29}$$

and for any $w \in B_1(0^N)$ there exists $x \in B_1(p)$ such that $|w - u(x)| \le \varepsilon$.

Sketch of proof. We outline a proof based on a contradiction argument, building on top of the convergence and stability of Sobolev functions along sequences of $\mathrm{RCD}(K, N)$ metric measure spaces converging in the pointed measured Gromov-Hausdorff sense developed in [57, 7, 8]. If the ambient space is a smooth Riemannian manifold, then the original argument from [29] can be made effective, i.e. it does not involve a contradiction argument and the dependence of δ from ε can be made explicit. See [97] for a detailed argument.

We suppose by contradiction that the statement fails. Then there exist a sequence of $\mathrm{RCD}(-1/n, N)$ metric measure spaces $(X_n, \mathsf{d}_n, \mathscr{H}^N)$, reference points $p_n \in X_n$ and harmonic functions $u^n : B_3(p_n) \to \mathbb{R}^N$ such that $u^n(p_n) = 0^N$ and

$$\sum_{i,j} \fint_{B_2(p)} \left|\nabla u_i^n \cdot \nabla u_j^n - \delta_{ij}\right| \mathrm{d}\mathscr{H}^N \le \frac{1}{n}\,, \tag{6.30}$$

while either

$$||u^n(x) - u^n(y)| - \mathsf{d}(x,y)| > \varepsilon_0\,, \quad \text{for some } x, y \in B_1^n(p_n) \tag{6.31}$$

or there exists some $w_n \in B_1(0^N)$ such that for any $x_n \in B_1^n(p_n)$

$$|w_n - u(x_n)| \ge \varepsilon_0\,, \tag{6.32}$$

for some $\varepsilon_0 > 0$.

Notice that (6.30) is invariant under normalization of the reference measure. Up to the extraction of a subsequence we can assume that

$$(X_n, \mathsf{d}_n, \mathscr{H}^N/\mathscr{H}^N(B_1^n(p_n)), p_n) \to (Y, \mathsf{d}_y, \mathfrak{m}_Y, p) \tag{6.33}$$

in the pointed measured Gromov-Hausdorff topology for some $\mathrm{RCD}(0, N)$ metric measure space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$. By compactness and stability of Sobolev functions, up to the extraction of a further subsequence that we do not relabel, the functions u_n uniformly converge to some limit function $u : B_3(p) \to \mathbb{R}^N$ with the following properties:

a) u is harmonic and $u(p) = 0^N$;
b) $\mathfrak{m}_Y$-a.e. on $B_2(p)$ it holds

$$|\nabla u_i| = 1\,, \quad \text{for any } i = 1, \dots, N, \quad \nabla u_i \cdot \nabla u_j = 0\,, \quad \text{for any } 1 \le i < j \le N\,. \tag{6.34}$$

Both statements can be proved with the tools developed in [57, 7, 8]. See [18, 19] for the detailed proofs. Statement b) follows from the lower semicontinuity:

$$\fint_{B_2(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}| \,\mathrm{d}\mathfrak{m}_Y \le \liminf_{n\to\infty} \fint_{B_2(p_n)} \left|\nabla u_i^n \cdot \nabla u_j^n - \delta_{ij}\right| \mathrm{d}\mathscr{H}^N = 0\,. \tag{6.35}$$

Given a) and b) a local version of the argument employed in the proof of the splitting Theorem 3.10 shows that $B_1(p)$ is isomorphic to the Euclidean ball $B_1(0^N) \subset R^N$ and $u : B_1(p) \to B_1(0^N)$ is an isometry, see [17] for the detailed argument. This is in contradiction with (6.31), (6.32). □

Lemma 6.11. *Let (X, d) be a metric space and let $E \subset X$. If $\varepsilon \le \varepsilon(N)$ and a map $u : E \to \mathbb{R}^k$ is a scale invariant ε-isometry on $B_r(x) \cap E$ for any $x \in E$ and any $0 < r < \mathrm{diam}(E)$, then u is $(1+\varepsilon)$-biLipschitz on E.*

Proof. By assumption

$$||u(x) - u(y)| - \mathsf{d}(x,y)| \le \varepsilon r\,, \tag{6.36}$$

for any $x \in E$ and any $y \in E$ such that $\mathsf{d}(x,y) \le r$. We choose $r := \mathsf{d}(x,y)$. Then

$$(1-\varepsilon)\mathsf{d}(x,y) \le |u(x) - u(y)| \le (1+\varepsilon)\mathsf{d}(x,y)\,, \tag{6.37}$$

for any $x, y \in E$.

□

6.3. **Volume convergence.** The proof of the volume convergence Theorem 5.5 is based on an ε-regularity theorem, to the effect that balls Gromov-Hausdorff close to Euclidean ones have volume close to the Euclidean one, and on a series of covering arguments.

Proposition 6.12. *Let $N \geq 1$ be fixed. For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N) > 0$ such that the following holds: if $(X, \mathsf{d}, \mathscr{H}^N)$ is an $\mathrm{RCD}(-\delta, N)$ metric measure space and*

$$\mathsf{d}_{\mathrm{GH}}\left(B_4(p), B_4(0^N)\right) \leq \delta\,, \tag{6.38}$$

then

$$\left|\mathscr{H}^N(B_1(p)) - \omega_N\right| \leq \varepsilon\,. \tag{6.39}$$

Basically all the arguments in the literature to prove Proposition 6.12 employ some *good* almost splitting functions, either harmonic functions or distance functions, to control the volume gap as in (6.39). The idea is that these good coordinates have small volume distortion, in integral sense, and moreover they are always Lipschitz. This is usually combined with some almost surjectivity type statement, to the effect that the image of the unit ball $B_1(p)$ has almost full measure inside the ball $B_1(0^N) \subset \mathbb{R}^N$. Then it is easy to conclude that (6.39) holds. See [38, 26, 45].

We sketch a different outline, avoiding the almost surjectivity and based on the construction of an approximate solution of the equations

$$\Delta r^2 = 2N\,, \quad |\nabla r| = 1\,. \tag{6.40}$$

The idea is that the existence of a solution to (6.40) would force the scale invariant volume ratio to be constant along scales. It is borrowed from the very recent [16].

Sketch of proof of Proposition 6.12. By the rectifiable structure Theorem 5.3 and the general theory of rectifiable metric spaces from [75],

$$\lim_{r\to 0}\frac{\mathscr{H}^N(B_r(x))}{\omega_N r^N} = 1\,, \quad \text{for } \mathscr{H}^N\text{-a.e. } x \in X\,. \tag{6.41}$$

In particular, we can find x arbitrarily close to p such that the density $\Theta_N(x)$ is 1.

Let $u : B_3(p) \to \mathbb{R}^N$ be the harmonic almost splitting maps constructed in Theorem 6.7. Arguing as in the proof of Proposition 6.9, we can find a point $x \in B_1(p)$ with $\mathsf{d}(x,p) < 1/100$, $\Theta_N(x) = 1$ and

$$\sup_{0<r<2} ⨍ |\nabla u_i \cdot \nabla u_j - \delta_{ij}|\,\mathrm{d}\mathscr{H}^N \leq f(\delta)\,, \tag{6.42}$$

with $f(\delta) \downarrow 0$ as $\delta \downarrow 0$. Up to subtracting a constant vector to u we can assume without loss of generality that $u(x) = 0$. Then we define

$$r^2 = \sum_{i=1}^N u_i^2\,. \tag{6.43}$$

We employ (6.42) to show that r is an approximate solution of (6.40) in integral sense at all scales $0 < r < 2$. Then we plug this information into the proof of the Bishop-Gromov monotonicity to deduce that the scale invariant volume ratio is almost constant between 0 and 2. As it converges to 1 when $r \downarrow 0$ by our choice of the base point with $\Theta_N(x) = 1$, it is very close to 1 at scale 1, which corresponds to (6.39). □

6.4. **ε-regularity and classical Reifenberg theorem.** The original proof of the topological manifold structure theorem for non collapsed Ricci limit spaces in [30] was based on Reifenberg's technique and the Euclidean almost regularity theorem.

Theorem 6.13. *Let $N \geq 1$ be fixed. Then there exists $\varepsilon(N)$ such that the following holds. If $(X, \mathsf{d}, \mathscr{H}^N)$ is an* $\mathrm{RCD}(-\varepsilon, N)$ *metric measure space $p \in X$ and*

$$\mathsf{d}_{\mathrm{GH}}\left(B_4(p), B_4(0^N)\right) \leq \varepsilon\,, \tag{6.44}$$

for some $\varepsilon \leq \varepsilon(N)$, then there exists a topological embedding $F : B_1(0^N) \to B_1(x)$ such that $F\left(B_1(0^N)\right) \supset B_{1-\Psi}(x)$, where $\Psi = \Psi(\varepsilon, N)$. Moreover, the maps F, F^{-1} are Hölder continuous with Hölder exponent $\alpha = 1 - \Psi$.

Proof. By volume convergence Theorem 5.5, if (6.44) holds then

$$\left|\mathscr{H}^N(B_3(p)) - 3^N\omega_N\right| \leq \delta\,, \tag{6.45}$$

with $\delta \to 0$ as $\varepsilon \to 0$. Then we can apply Bishop-Gromov volume monotonicity to deduce that

$$\left|\frac{\mathscr{H}^N(B_r(x))}{\omega_N r^N} - 1\right| \leq f(\delta, N)\,, \quad \text{for any } 0 < r < 2 \text{ and any } x \in B_1(p)\,, \tag{6.46}$$

for some function $f(\delta, N) \geq 0$ such that $f(\delta, N) \downarrow 0$ as $\delta \to 0$.
By the volume convergence Theorem 5.5, compactness and the almost Euclidean volume rigidity (cf. with Remark 5.14) it is easy to check that (6.46) forces

$$\mathsf{d}_{\mathrm{GH}}\left(B_r(x), B_r(0^N)\right) \leq g(\delta, N) r\,, \quad \text{for any } 0 < r < 1 \text{ and any } x \in B_1(p)\,, \tag{6.47}$$

for some function $g(\delta, N) \geq 0$ such that $g(\delta, N) \downarrow 0$ as $\delta \to 0$. The conclusion follows from Reifenberg's Theorem 6.16 for metric spaces below. □

We briefly digress on Reifenberg's theorem addressing to the lecture notes [82] for a detailed introduction to the classical theory and to some of the most recent developments.

Definition 6.14. Let $S \subset B_2(0) \subset \mathbb{R}^N$ be a closed set. We define the L^∞-Jones β-numbers

$$\beta_k^\infty(x, r) := r^{-1} \inf_L \mathsf{d}_H\left(S \cap B_r(x), L \cap B_r(x)\right)\,, \tag{6.48}$$

where the infimum runs among all k-dimensional affine subspaces $L \subset \mathbb{R}^N$.
We say that S satisfies the δ-Reifenberg condition provided

$$\beta_k^\infty(x, r) \leq \delta\,, \quad \text{for any } x \in S \text{ and any } r > 0 \text{ such that } B_r(x) \subset B_2(0)\,. \tag{6.49}$$

We can state the classical Reifenberg theorem from [89].

Theorem 6.15. *Let $S \subset B_2(0) \subset \mathbb{R}^n$ satisfy the δ-Reifenberg condition. Then for every $0 < \alpha < 1$, if $\delta < \delta(n, \alpha)$ then there exists a map $\varphi : B_1(0^n) \cap S \to B_1(0^k)$ which is a α-biHölder map, precisely*

$$\frac{1}{2}\left|x - y\right|^{1+\alpha} \leq \left|\varphi(x) - \varphi(y)\right| \leq 2\left|x - y\right|^{1-\alpha}\,. \tag{6.50}$$

Cheeger-Colding in [30, Appendix A] achieved a far reaching generalization of Reifenberg's Theorem 6.15, valid for general complete metric spaces.

Theorem 6.16. *There exists $\varepsilon(N) > 0$ with the following property. Let (X, d) be a complete metric space. Assume that for some $x \in X$ and some $\varepsilon \leq \varepsilon(N)$ it holds that*

$$\mathsf{d}_{GH}\left(B_r(x'), B_r(0^N)\right) \leq \varepsilon r\,, \tag{6.51}$$

for any $x' \in B_1(z)$ and any $0 < r < 1 - \mathsf{d}(x', x)$. Then there exists a topological embedding $F : B_1(0^N) \to B_1(x)$ such that $F\left(B_1(0^N)\right) \supset B_{1-\Psi}(x)$, where $\Psi = \Psi(\varepsilon, N)$. Moreover, the maps F, F^{-1} are Hölder continuous with Hölder exponent $\alpha = 1 - \Psi$. If (X, d) is isometric to a smooth Riemannian manifold, then F can be taken to be a smooth embedding.

In the recent [34], Cheeger-Jiang-Naber proved a *canonical* Reifenberg theorem for spaces with lower Ricci curvature bounds, see Theorem 7.10 therein and the subsequent [17] for the minor modifications needed for the RCD setting. By canonical, it is meant that the biHölder chart is not constructed with the iteration procedure going into the proof of Theorem 6.16. Instead, the chart is the solution of an equation.

Theorem 6.17. *Let $N \geq 1$ be fixed. For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, N) > 0$ such that the following holds. If $(X, \mathsf{d}, \mathscr{H}^N)$ is an $\mathrm{RCD}(-\delta, N)$ metric measure space and for some $p \in X$ it holds*

$$\mathsf{d}_{GH}(B_4(p), B_4(0^N)) \leq \delta \,, \tag{6.52}$$

then the map $u : B_3(p) \to \mathbb{R}^N$ obtained in Theorem 6.7 satisfies

$$(1-\varepsilon)\mathsf{d}^{1+\varepsilon}(x,y) \leq |u(x) - u(y)| \leq (1+\varepsilon)\mathsf{d}(x,y) \,, \quad \text{for any } x, y \in B_1(p) \,. \tag{6.53}$$

Moreover, if (X, d) is isometric to a smooth Riemannian manifold, then u is a smooth diffeomorphism.

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