Lecture 4: first and second variation of the area on RCD spaces

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Outline

Questions

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Do area minimizing hypersurfaces in RCD spaces have vanishing mean curvature? Are isoperimetric surfaces CMC?

Does the lower Ricci curvature bound affect the second variation of the area in RCD spaces?

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Ricci bounds and minimal hypersurfaces, I

(M^n, g) smooth Riemannian manifold; $\Sigma^{n-1} \subset M$ smooth and compact codimension one hypersurface.

X compactly supported, smooth vector field; denote its flow by $\Phi_t : M \times (-\varepsilon, \varepsilon) \to M$. Then we have the first variation formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_\Sigma \mathrm{div}_\Sigma X \,\mathrm{d}\,\mathcal{H}^{n-1} = -\int_\Sigma H \cdot X \,\mathrm{d}\,\mathcal{H}^{n-1} .$$

Corollary

If Σ minimizes the area among compactly supported perturbations then it is minimal, i.e. $H = 0$.

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Ricci bounds and minimaly hypersurfaces, II

Let Σ be minimal and two-sided with unit normal ν. We can compute the second variation of the area for vector fields X such that $X = f\nu$ along Σ:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_\Sigma \left[|\nabla_\Sigma f|^2 - \left(|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu)\right) f^2\right] \mathrm{d}\,\mathcal{H}^{n-1}\,.$$

By plugging $f \equiv 1$ (equidistant variation) in the second variation formula we get

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Bad examples

Example

Let D be a flat two dimensional disk with boundary C. Let $\tilde{D}$ be the metric space obtained by doubling D along the boundary.

In [Otsu-Shioya, JDG '94] an example of 2d convex surface with a dense set of singular points is constructed.

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- The metric space $(\tilde{D}, \mathsf{d}_{\tilde{D}}, \mathcal{H}^2)$ is $\mathrm{RCD}(0, 2)$. There is singular distributional Gaussian curvature along the copy of C;
- the area functional is not differentiable along normal variations.

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In [Otsu-Shioya, *JDG* '94] an example of $2d$ convex surface with a dense set of singular points is constructed.

Comments

- Any $\mathrm{RCD}(K,N)$ space X is isomorphic to a perimeter minimizing boundary in $X \times \mathbb{R}$;
- any $\mathrm{RCD}(N-2,N-1)$ space X is isomorphic to an isoperimetric set in the cone $C(X)$;
- the classical regularity theory in GMT does not make sense in this setting;
- the regularity theory for flows of vector fields under lower Ricci bounds ([Colding-Naber, *Ann. of Math.* '11], [Kapovitch-Wilking '12], [Bruè-S., *CPAM* '18], [Deng '20], ...) seems not sufficiently developed for permitting a first variation formula for the perimeter.

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Laplacian comparison for perimeter minimizers

For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)}\tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)}\tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

Theorem (Mondino-S. '21)

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a local perimeter minimizer. Let $\mathsf{d}_{\overline{E}} : X \setminus \overline{E} \to [0, \infty)$ be the distance function from $\overline{E}$. Then

$$\Delta \mathsf{d}_{\overline{E}} \leq \tau_{K,N} \circ \mathsf{d}_{\overline{E}} \quad \text{on } X \setminus \overline{E}.$$

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First and second variation RCD

Daniele Semola

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$$s_{k,\lambda}(r) := \cos_k(r) - \lambda \sin_k(r)\,,$$

$$\cos_k'' + k \cos_k = 0\,, \quad \cos_k(0) = 1\,, \quad \cos_k'(0) = 0\,,$$
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Theorem (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space and $E \subset X$ be an isoperimetric region. Then, there exists $c \in \mathbb{R}$ such that

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The $K = 0$ case

If $K = 0$ (nonnegative Ricci) then the bounds take the nicer form.

Corollary

If $(X, \mathsf{d}, \mathcal{H}^N)$ is $\mathrm{RCD}(0, N)$ and $E \subset X$ is isoperimetric, there exists $c \in [0, \infty)$ such that

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If the ambient is smooth Riemannian and ∂E is smooth, then c must be equal to the (constant) mean curvature of ∂E.

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Comments

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $\mathrm{RCD}(K, N)$ spaces. They can be understood distributionally;
- the bounds are sharp and attained on the model spaces;
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First and second variation of the perimeter

Corollary

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(0, N)$ space and let $E \subset X$ be isoperimetric. Then, denoting by E_t the t-enlargement of E,

$$\mathrm{Per}(E_t) \le \mathrm{Per}(E)\left(1 + \frac{ct}{N-1}\right)^{N-1}, \quad \text{for any } t \ge 0 .$$

Proof.

Apply Gauss-Green on a tubular neighbourhood of ∂E and ODEs comparison, taking into account the Laplacian comparison. □

The estimate is sharp. It replaces the classical computation with the second variation formula, to some extent.

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Classical case: smooth minimal hypersurface

$\partial E = \Sigma^{N-1} \subset X^N$ smooth and minimal inside a smooth Riemannian manifold: statement goes back at least to [Wu, *Acta Math.* '79]. The bound was understood in the viscosity sense.

- Along the minimal boundary the Laplacian of the distance equals the (vanishing) mean curvature;
- the information is propagated along minimizing geodesics up to the cut-locus via the Jacobi fields computation

$$\frac{d}{dt}\Delta d_\Sigma(\gamma(t)) + \|\mathrm{Hess}\, d_\Sigma(\gamma(t))\|^2_{HS} + \mathrm{Ric}_g(\nabla d_\Sigma, \nabla d_\Sigma) = 0$$

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Classical case: area minimizing hypersurface

For general area minimizing hypersurfaces (currents, sets of finite perimeter) the classical argument is due to [Gromov '81].

- Minimality needed only at foot-points F_P of geodesics γ_P on the boundary $\Sigma = \partial E$;
- for a.e. point at the foot-point on the boundary all the blow-ups of the current are contained in a half-space;
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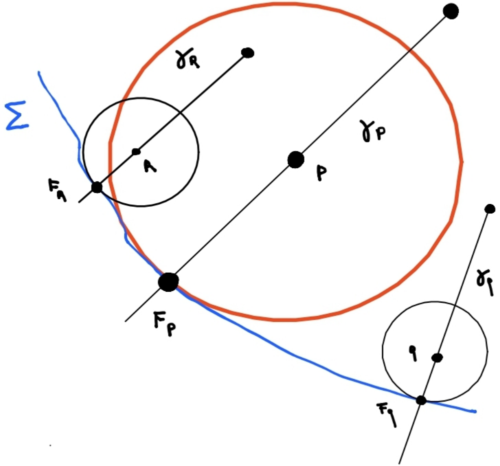
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From a-dimensional to dimensional bounds

By the localization technique ([Cavalletti-Mondino '15, '18], after [Klartag '14]) the Laplacian of any distance function on an $\mathrm{RCD}(K,N)$ space verifies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta \mathrm{d}(\gamma(t)) + \frac{1}{N-1}\left(\Delta \mathrm{d}(\gamma(t))\right)^2 \le -K\,,$$

along minimizing geodesics such that $\mathrm{d}(\gamma(t)) = t+\alpha$. Moreover, the singular contribution has negative sign.

Corollary

Let $(X,\mathrm{d},\mathcal{H}^N)$ be an $\mathrm{RCD}(K,N)$ space and $E\subset X$. Then

$$\Delta \mathrm{d}_{\overline{E}} \le \tau_{K,N}\circ \mathrm{d}_{\overline{E}} \quad \text{on } X\setminus\overline{E}$$

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Comparison with previous literature

Inspired by [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on $\mathbb{R}^n$ and a sketch in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

Remark

It exploits the duality between viscous, distributional and variational interpretation of Laplacian bounds.

- In [Caffarelli-Cordoba '93] proof via the viscosity theory using comparison with quadratic polynomials and the affine structure;
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A PDE principle, I

Theorem

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ space and let $\varphi : X \to \mathbb{R}$ be such that $\Delta\varphi \le 0$ on $\Omega \subset X$. Then

$$Q_t^p\varphi(x) := \inf_{y \in X}\left\{\varphi(y) + \frac{\mathsf{d}^p(x, y)}{pt^{p-1}}\right\}$$

verifies $\Delta Q_t^p\varphi \le 0$ in the region where the infumum is attained on Ω, for any $1 < p < \infty$ and for any $t > 0$.

Hopf-Lax semigroup $Q_t^p\varphi$ solves Hamilton-Jacobi equation:

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A PDE principle, I

Theorem

Let $(X, \mathrm{d}, \mathrm{m})$ be an $\mathrm{RCD}(0, N)$ space and let $\varphi : X \to \mathbb{R}$ be such that $\Delta\varphi \leq 0$ on $\Omega \subset X$. Then

$$\mathcal{Q}_t^p \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{\mathrm{d}^p(x,y)}{p t^{p-1}} \right\}$$

verifies $\Delta \mathcal{Q}_t^p \varphi \leq 0$ in the region where the infumum is attained on Ω, for any $1 < p < \infty$ and for any $t > 0$.

Remark

Hopf-Lax semigroup $\mathcal{Q}_t^p \varphi$ solves Hamilton-Jacobi equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{Q}_t^p \varphi + \frac{1}{q} |\nabla \mathcal{Q}_t^p \varphi|^q = 0\,, \quad (1/p + 1/q = 1)\,.$$

A PDE principle, II

For smooth Riemannian manifolds the PDE principle follows from a computation with Jacobi fields/second variation of the arc length, see [Andrews-Clutterbuck, *Anal. PDE* '13] (and [Petrunin '97]) for morally analogous estimates.

Remark

A smooth Riemannian manifold has nonnegative Ricci if and only the PDE principle holds.

In the RCD setting, the proof follows in almost elementary way from the so-called Kuwada duality with the heat flow:

$$P_s \mathcal{Q}_t^p \varphi \le \mathcal{Q}_t^p P_s \varphi \quad \text{for any } s \ge 0 \,,$$

after [Kuwada, *JFA* '10], [Ambrosio-Gigli-Savaré, *Ann. of Prob.* '14].

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Laplacian comparison: strategy of the proof

- Suppose $K = 0$, E perimeter minimizing and that super-harmonicity of d_E fails. We find a lower supporting function φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- We consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - \mathrm{d}(y, z)\} .$$

- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_E and $\tilde{\varphi} \le \mathrm{d}_E$.
- $\tilde{\varphi}$ is a distance-like function and $\Delta\tilde{\varphi} > \varepsilon > 0$ near to x_E, by the PDE principle.
- We cut ∂E along level sets of $\tilde{\varphi}$, apply Gauss-Green and contradict the area minimality.

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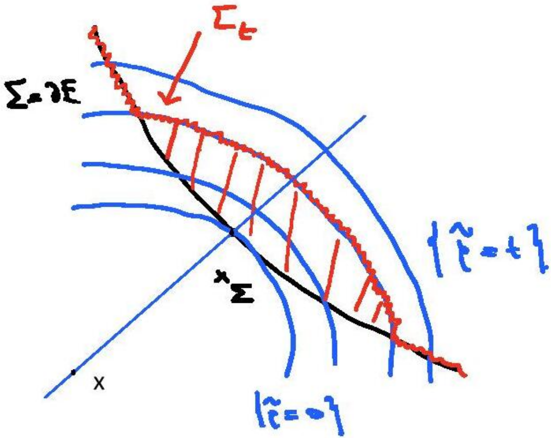
Strategy of the proof, II



The isoperimetric case

Only volume fixing perturbations are admissible.

- If $K = 0$ it is sufficient to find a *barrier* $c \in \mathbb{R}$ such that $\Delta \mathrm{d}_{\overline{E}} \leq c$ outside from E and $\Delta(-\mathrm{d}_{X \setminus \overline{E}}) \geq c$ inside E.
- If no such barrier exists we find points $x \in X \setminus E$ and $y \in E$, a regular lower touching function φ for $\mathrm{d}_{\overline{E}}$ at x, an upper touching function ψ for $-\mathrm{d}_{X \setminus \overline{E}}$ at y, such that

$$\Delta\varphi(x) > \Delta\psi(y) .$$

- We play the same game as before simultaneously with φ and ψ. By continuity, there is a volume fixing perturbation with strictly smaller perimeter, contradiction.

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Thank you for your attention!