

Lecture 4: first and second variation of the area on RCD spaces

Daniele Semola

Mathematical Institute, University of Oxford

Daniele.Semola@maths.ox.ac.uk

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Do area minimizing hypersurfaces in RCD spaces have vanishing mean curvature? Are isoperimetric surfaces CMC?

Does the mean Ricci curvature bound affect the second variation of volume in RCD spaces?

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Do **area minimizing** hypersurfaces in RCD spaces have **vanishing mean curvature**? Are **isoperimetric** surfaces **CMC**?

Does the lower Ricci curvature bound affect the second variation of the area in RCD spaces?

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Motivations

- Understand Curvature, [Gromov '19];
- GMT on singular spaces as a new tool for classical GMT.

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(M^n, g) smooth Riemannian manifold; $\Sigma^{n-1} \subset M$ smooth and compact codimension one hypersurface.

X compactly supported, smooth vector field; denote its flow by $\Phi_t : M \times (-\varepsilon, \varepsilon) \rightarrow M$. Then we have the first variation formula

$$\frac{d}{dt} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} \operatorname{div}_{\Sigma} X \, d\mathcal{H}^{n-1} = - \int_{\Sigma} H \cdot X \, d\mathcal{H}^{n-1}.$$

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If Σ minimizes the area among compactly supported perturbations then it is minimal, i.e. $H \equiv 0$.

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Corollary

If Σ minimizes the area among compactly supported perturbations then it is minimal, i.e. $H \equiv 0$.

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Let Σ be minimal and two-sided with unit normal ν . We can compute the second variation of the area for vector fields X such that $X = f\nu$ along Σ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} [|\nabla_{\Sigma} f|^2 - (|\mathbb{H}|^2 + \text{Ric}(\nu, \nu)) f^2] d\mathcal{H}^{n-1}.$$

By plugging $f \equiv 1$ (equidistant variation) in the second variation formula we get

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Sigma_t) = - \int_{\Sigma} (|\mathbb{H}|^2 + \text{Ric}(\nu, \nu)) d\mathcal{H}^{n-1}.$$

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$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{n-1}(\Phi_t(\Sigma)) = \int_{\Sigma} [|\nabla_{\Sigma} f|^2 - (|\mathbf{II}|^2 + \text{Ric}(\nu, \nu)) f^2] d\mathcal{H}^{n-1}.$$

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Example

Let D be a **flat** two dimensional **disk** with boundary C . Let \tilde{D} be the metric space obtained by **doubling** D along the boundary.

- The metric space $(\tilde{D}, d_g, \mathcal{H}^2)$ is κ RCD(0, 2). There is singular distributional Gaussian curvature along the copy of C .

• The metric space $(\tilde{D}, d_g, \mathcal{H}^2)$ is not κ RCD(0, 2) in the sense of [Ambrosio et al., 2014].

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In [Otsu-Shioya, JDG '94] an example of 2d convex surface with a dense set of singular points is constructed.

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Example

In [Otsu-Shioya, *JDG* '94] an example of $2d$ convex surface with a **dense** set of **singular points** is constructed.

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- Any $\text{RCD}(K, N)$ space X is isomorphic to a perimeter minimizing boundary in $X \times \mathbb{R}$;
- any $\text{RCD}(N - 2, N - 1)$ space X is isomorphic to an isoperimetric set in the cone $C(X)$;
- the classical regularity theory in CDE spaces is more subtle in this setting;
- the regularity theory for spaces of vanishing Ricci is not known (RCD bounds \Rightarrow vanishing Ricci \Rightarrow CDE);
- the regularity theory for spaces of vanishing Ricci is not sufficiently developed for proving the isoperimetric formula for the perimeter.

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- the classical regularity theory in GMT does not make sense in this setting;
- the regularity theory for flows of vector fields under lower Ricci bounds ([Colding-Naber, *Ann. of Math.* '11], [Kapovitch-Wilking '12], [Bruè-S., *CPAM* '18], [Deng '20], ...) seems not sufficiently developed for permitting a first variation formula for the perimeter.

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Laplacian comparison for perimeter minimizers

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For $K \in \mathbb{R}$ and $1 \leq N < \infty$ let

- $\tau_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x)$ if $K > 0$;
- $\tau_{0,N} := 0$;
- $\tau_{K,N} := \sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x)$ if $K < 0$.

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Let (X, d, \mathcal{H}^N) be an RCD(K, N) metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a local perimeter minimizer. Let $d_E : X \setminus \bar{E} \rightarrow [0, \infty)$ be the distance function from \bar{E} . Then

$$\Delta d_E \leq \tau_{K,N} \circ d_E \quad \text{on } X \setminus \bar{E}.$$

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Theorem (Mondino-S. '21)

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ metric measure space. Let $E \subset X$ be a set of locally finite perimeter and assume that it is a *local perimeter minimizer*. Let $d_{\bar{E}} : X \setminus \bar{E} \rightarrow [0, \infty)$ be the *distance function* from \bar{E} . Then

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For $k, \lambda \in \mathbb{R}$ let

$$s_{k,\lambda}(r) := \cos_k(r) - \lambda \sin_k(r),$$

$$\cos_k'' + k \cos_k = 0, \quad \cos_k(0) = 1, \quad \cos_k'(0) = 0,$$

$$\sin_k'' + k \sin_k = 0, \quad \sin_k(0) = 0, \quad \sin_k'(0) = 1.$$

Let (X, d, \mathcal{H}^M) be an $\text{RCD}(K, N)$ space and $E \subset X$ be an isoperimetric region. Then, there exists $c \in \mathbb{R}$ such that

$$\Delta d_E \geq -(N-1) \frac{s'_{N-1, N-1} \circ (-d_E)}{s_{N-1, N-1} \circ (-d_E)} \quad \text{on } E,$$

$$\Delta d_E \leq (N-1) \frac{s'_{N-1, -N-1} \circ d_E}{s_{N-1, -N-1} \circ d_E} \quad \text{on } X \setminus E.$$

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Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ space and $E \subset X$ be an isoperimetric region. Then, there exists $c \in \mathbb{R}$ such that

$$\Delta d_E \geq -(N-1) \frac{s'_{K, \lambda(K)} \circ (-d_E)}{s_{K, \lambda(K)} \circ (-d_E)} \quad \text{on } E,$$

$$\Delta d_E \leq (N-1) \frac{s'_{-K, -\lambda(K)} \circ d_E}{s_{-K, -\lambda(K)} \circ d_E} \quad \text{on } X \setminus E.$$

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$$\sin_k'' + k \sin_k = 0, \quad \sin_k(0) = 0, \quad \sin_k'(0) = 1.$$

Theorem (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let (X, d, \mathcal{H}^N) be an RCD(K, N) space and $E \subset X$ be an isoperimetric region. Then, there exists $c \in \mathbb{R}$ such that

$$\Delta d_{\bar{E}} \geq -(N-1) \frac{s'_{\frac{K}{N-1}, \frac{c}{N-1}} \circ (-d_{\bar{E}})}{s_{\frac{K}{N-1}, \frac{c}{N-1}} \circ (-d_{\bar{E}})} \quad \text{on } E,$$

$$\Delta d_{\bar{E}} \leq (N-1) \frac{s'_{\frac{K}{N-1}, -\frac{c}{N-1}} \circ d_{\bar{E}}}{s_{\frac{K}{N-1}, -\frac{c}{N-1}} \circ d_{\bar{E}}} \quad \text{on } X \setminus \bar{E}.$$

The $K = 0$ case

If $K = 0$ (nonnegative Ricci) then the bounds take the nicer form.

If (X, d, \mathcal{H}^N) is $\text{RCD}(0, N)$ and $E \subset X$ is isoperimetric, there exists $c \in [0, \infty)$ such that

$$\Delta d_E \leq \frac{c}{1 + \frac{c}{N-1} d_E}, \quad \text{on } X \setminus \bar{E},$$

$$\Delta(-d_{X \setminus E}) \geq \frac{c}{1 - \frac{c}{N-1} d_{X \setminus E}}, \quad \text{on } E.$$

If the ambient is smooth Riemannian and $\text{Ric} \geq 0$ is smooth, then c can be chosen to be the constant $\frac{1}{2} \max_{X \setminus \bar{E}} |\text{Ric}|$.

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If (X, d, \mathcal{H}^N) is $\text{RCD}(0, N)$ and $E \subset X$ is isoperimetric, then

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The $K = 0$ case

If $K = 0$ (nonnegative Ricci) then the bounds take the nicer form.

Corollary

If (X, d, \mathcal{H}^N) is $\text{RCD}(0, N)$ and $E \subset X$ is isoperimetric, there exists $c \in [0, \infty)$ such that

$$\Delta d_{\bar{E}} \leq \frac{c}{1 + \frac{c}{N-1} d_{\bar{E}}}, \quad \text{on } X \setminus \bar{E},$$

$$\Delta(-d_{X \setminus E}) \geq \frac{c}{1 - \frac{c}{N-1} d_{X \setminus E}}, \quad \text{on } E.$$

If the ambient is smooth Riemannian and ∂E is smooth, then c must be equal to the (constant) mean curvature of ∂E .

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Remark

If the ambient is smooth Riemannian and ∂E is smooth, then c must be equal to the (constant) **mean curvature** of ∂E .

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- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $RCD(K, N)$ spaces. They can be understood distributionally;
- the bounds are sharp and achieved on the radial coordinate; in fact, the bounds imply that the λ -weighted Laplacian of the distance function is bounded above by $\lambda + K$;
- if a non smooth boundary inside a smooth Riemannian manifold, then they imply sharp isoperimetric inequalities for the boundary;
- it would be interesting to talk about the λ -weighted Laplacian on bounded domains.

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- The distance function is not smooth even on smooth Riemannian manifolds;
 - the bounds make perfectly sense on $RCD(K, N)$ spaces. They can be understood **distributionally**;
 - the bounds are **sharp** and attained on the model spaces;
 - on \mathbb{R}^N the bounds imply that ∂E is a **viscosity solution** of the **minimal surfaces/constant mean curvature equation** [Savin, *Comm. PDEs '07*];
 - if E has smooth boundary inside a smooth Riemannian manifold, then they imply **vanishing/constant mean curvature** for the boundary;
 - no need to talk about **mean curvature** of the area minimizing boundary.

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First and second variation of the perimeter

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Let (X, d, \mathcal{H}^N) be an RCD(0, N) space and let $E \subset X$ be isoperimetric. Then, denoting by E_t the t -enlargement of E ,

$$\text{Per}(E_t) \leq \text{Per}(E) \left(1 + \frac{ct}{N-1}\right)^{N-1}, \quad \text{for any } t \geq 0.$$

• The result is a natural generalization of the classical isoperimetric inequality in Euclidean space, which is obtained by taking $N=2$ and $c=2$.

The result is proved in the paper: [D. Semola, First and second variation of the perimeter in RCD spaces, *Journal of Functional Analysis* 370 \(2020\) 109007](#).

First and second variation of the perimeter

Corollary

Let (X, d, \mathcal{H}^N) be an RCD(0, N) space and let $E \subset X$ be isoperimetric. Then, denoting by E_t the t -enlargement of E ,

$$\text{Per}(E_t) \leq \text{Per}(E) \left(1 + \frac{ct}{N-1}\right)^{N-1}, \quad \text{for any } t \geq 0.$$

Apply Gauss-Green on a tubular neighbourhood of ∂E and ODEs comparison, taking into account the Laplacian comparison. \square

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Proof.

Apply [Gauss-Green](#) on a tubular neighbourhood of ∂E and [ODEs comparison](#), taking into account the Laplacian comparison. \square

The estimate is sharp. It replaces the classical computation with the second variation formula, to some extent.

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$\partial E = \Sigma^{N-1} \subset X^N$ smooth and minimal inside a smooth Riemannian manifold: statement goes back at least to [Wu, *Acta Math.* '79]. The bound was understood in the viscosity sense.

- Along the minimal boundary the Laplacian of the distance equals the (vanishing) mean curvature;
- the information is propagated along minimizing geodesics up to the cut-locus via the local PDE comparison.

$\Delta d(x, \cdot) = 0$ on ∂E and $\Delta d(x, \cdot) \leq 0$ on ∂E and $\Delta d(x, \cdot) \geq 0$ on ∂E .

is the distributional Laplacian. The argument used by Wu is the following:

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- Along the minimal boundary the Laplacian of the distance equals the (vanishing) mean curvature;
- the information is propagated along minimizing geodesics up to the cut-locus via the Jacobi fields computation

$$\frac{d}{dt} \Delta d_E(\gamma(t)) + \|\text{Hess} d_E(\gamma(t))\|_{\text{HS}}^2 + \text{Ric}_{\gamma(t)}(\nabla d_E, \nabla d_E) = 0.$$

and Riccati comparison for ODEs;

and the comparison of the Laplacian of the distance

with the Laplacian of the distance in the Euclidean space

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- the contribution coming from the singularities of the distance function has the right sign.

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For general **area minimizing** hypersurfaces (currents, sets of finite perimeter) the classical argument is due to [Gromov '81].

- Minimality needed only at foot-points F_p of geodesics γ_p on the boundary $\Sigma = \partial E$;

- for a p -point at the top point on the boundary of the hemisphere of the curved cap, the cap is a half-sphere;

- by the isoperimetric inequality, the area minimizing boundary is only π in a neighborhood of these points;

- this is the smooth argument for the cap.

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- Minimality needed only at foot-points F_p of geodesics γ_p on the boundary $\Sigma = \partial E$;
- for a.e. point at the foot-point on the boundary all the blow-ups of the current are contained in a half-space;
- the half-space is tangent to the boundary at the foot-point;
- the half-space is the supporting half-space of the current;
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- for a.e. point at the **foot-point** on the boundary all the blow-ups of the current are contained in a half-space;
- by **Almgren's regularity theorem** the area minimizing boundary is **smooth** in a neighbourhood of these points;
- then the smooth argument carries over.

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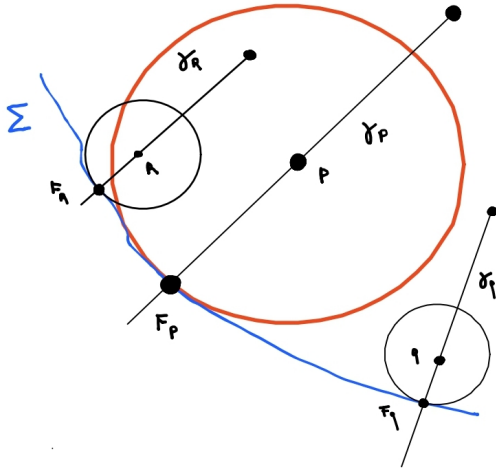
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From a-dimensional to dimensional bounds

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By the **localization technique** ([Cavalletti-Mondino '15, '18], after [Klartag '14]) the Laplacian of any distance function on an $\text{RCD}(K, N)$ space verifies

$$\frac{d}{dt} \Delta d(\gamma(t)) + \frac{1}{N-1} (\Delta d(\gamma(t)))^2 \leq -K,$$

along minimizing geodesics such that $d(\gamma(t)) = t + \alpha$. Moreover, the **singular contribution** has negative sign.

Let (X, d, \mathcal{H}^M) be an $\text{RCD}(K, N)$ space and $E \subset X$. Then

$$\Delta d_E \leq \tau_{K, N} \circ d_E \quad \text{on } X \setminus E$$

if and only if

$$\Delta d_E \leq -K d_E \quad \text{on } X \setminus E.$$

From a-dimensional to dimensional bounds

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Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ space and $E \subset X$. Then

$$\Delta d_E \leq \kappa_{K, N} \circ d_E \quad \text{on } X \setminus E$$

if and only if

$$\Delta d_E \leq -K d_E \quad \text{on } X \setminus E.$$

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along minimizing geodesics such that $d(\gamma(t)) = t + \alpha$. Moreover, the **singular contribution** has negative sign.

Corollary

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ space and $E \subset X$. Then

$$\Delta d_{\bar{E}} \leq \tau_{K,N} \circ d_{\bar{E}} \quad \text{on } X \setminus \bar{E}$$

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Comparison with previous literature

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Inspired by [Caffarelli-Cordoba, *Diff. Int. Eq.* '93] on \mathbb{R}^n and a sketch in [Petrunin, *E.R.A.* '03] for Alexandrov spaces.

It exploits the duality between viscous, distributional and variational interpretation of Laplacian bounds.

Comparison with previous literature

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Remark

It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- In [Caffarelli-Cordoba '93] proof via the viscosity theory, using comparison with quadratic polynomials and the affine structure;

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Remark

It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- In [Caffarelli-Cordoba '93] proof via the viscosity theory, using comparison with quadratic polynomials and the affine structure;
- in [Petrunin '03] (cf. also with [Cabré '87]) quadratic polynomials are avoided but the proof relies on parallel transport and the second variation of the arc length;

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It exploits the duality between **viscous**, **distributional** and **variational** interpretation of Laplacian bounds.

- In [Caffarelli-Cordoba '93] proof via the viscosity theory, using comparison with **quadratic polynomials** and the **affine structure**;
- in [Petrunin '03] (cf. also with [Cabré '97]) quadratic polynomials are avoided but the proof relies on **parallel transport** and the **second variation** of the arc length;
- we completely **avoid** the classical **regularity theory** for area minimizers and the use of **parallel transport**.

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Let (X, d, m) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be such that $\Delta\varphi \leq 0$ on $\Omega \subset X$. Then

$$Q_t^p \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{p t^{p-1}} \right\}$$

verifies $\Delta Q_t^p \varphi \leq 0$ in the region where the infimum is attained on Ω , for any $1 < p < \infty$ and for any $t > 0$.

Particular case: gradient of φ solves Hamilton-Jacobi equation:

$$\frac{1}{2} Q_t^2 \varphi - \frac{1}{2} \nabla Q_t^2 \varphi^2 = 0, \quad Q_t^2 \varphi + \frac{1}{2} Q_t^2 \varphi = 0$$

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Let (X, d, m) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be such that $\Delta\varphi \leq 0$ on $\Omega \subset X$. Then

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Hopf-Lax semigroup $Q_t^p \varphi$ solves Hamilton-Jacobi equation:

$$\frac{\partial}{\partial t} Q_t^p \varphi + \frac{1}{p} |\nabla Q_t^p \varphi|^p = 0, \quad (1/p + 1/q = 1)$$

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Let (X, d, m) be an $\text{RCD}(0, N)$ space and let $\varphi : X \rightarrow \mathbb{R}$ be such that $\Delta\varphi \leq 0$ on $\Omega \subset X$. Then

$$\mathcal{Q}_t^p \varphi(x) := \inf_{y \in X} \left\{ \varphi(y) + \frac{d^p(x, y)}{pt^{p-1}} \right\}$$

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For smooth Riemannian manifolds the PDE principle follows from a computation with Jacobi fields/second variation of the arc length, see [Andrews-Clutterbuck, *Anal. PDE* '13] (and [Petrunin '97]) for morally analogous estimates.

A smooth Riemannian manifold has nonnegative Ricci if and only if the PDE principle holds.

In the RCD setting, the proof follows in almost elementary way from the so-called Kuwada duality with the heat flow:

$$P_s Q_t^p \varphi \leq Q_t^p P_s \varphi \quad \text{for any } s \geq 0,$$

after [Kuwada, *JFA* '10], [Ambrosio-Gigli-Savaré, *Ann. of Prob.* '14].

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- Suppose $K = 0$, E perimeter minimizing and that super-harmonicity of d_E fails. We find a lower supporting function φ for d_E with strictly positive Laplacian at some $x \in X \setminus \partial E$.
- We consider the transform

$$\tilde{\varphi}(y) = \max_z \{\varphi(z) - d(y, z)\}.$$

- $\tilde{\varphi}$ coincides with d_E along ∂E (thanks to the fact that $\varphi \geq d_E$ and $\partial E \subseteq \partial \tilde{\varphi}$).
- The distance function d_E and the $\tilde{\varphi} \geq d_E$ agree to first order along ∂E .
- The $\tilde{\varphi}$ is a strong lower support of d_E along ∂E and we control the area minimality.

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- $\tilde{\varphi}$ coincides with d_E along a minimizing geodesic from x to x_Σ and $\tilde{\varphi} \leq d_E$.
- $\tilde{\varphi}$ is a **distance-like** function and $\Delta \tilde{\varphi} > \varepsilon > 0$ near to x_Σ , by the PDE principle.
- We **cut** ∂E along level sets of $\tilde{\varphi}$, apply **Gauss-Green** and contradict the area minimality.

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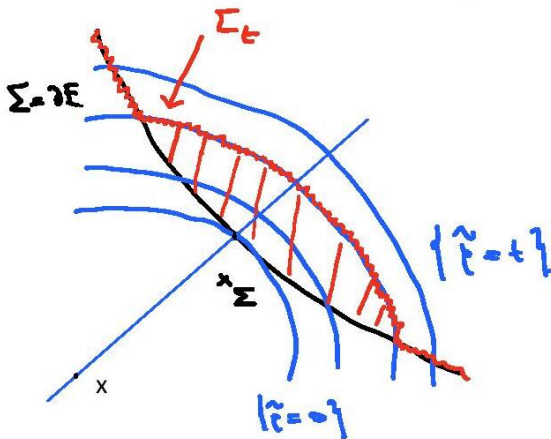
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Only volume fixing perturbations are admissible.

- If $K = 0$ it is sufficient to find a barrier $c \in \mathbb{R}$ such that $\Delta d_E \leq c$ outside from E and $\Delta(-d_{X \setminus E}) \geq c$ inside E .
- If no such barrier exists we find points $x \in X \setminus E$ and $y \in E$ such that $\Delta d_E(x) > c$ and $\Delta(-d_{X \setminus E})(y) < c$. We consider the function $f = -d_{X \setminus E} + d_E$ such that

$$\Delta f(x) > \Delta f(y)$$

• We play the same game as before simultaneously with c and $-c$. In particular, there is a volume fixing perturbation with strictly smaller perimeter contribution.

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- If no such barrier exists we find points $x \in X \setminus E$ and $y \in E$, a regular lower touching function φ for d_E at x , an upper touching function ψ for $-d_{X \setminus E}$ at y , such that

$$\Delta\varphi(x) > \Delta\psi(y).$$

Therefore, the volume variation is positive (independently from ν and μ).
Consequently, there are no volume fixing perturbations and the isoperimetric inequality is not satisfied.

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Only volume fixing perturbations are admissible.

- If $K = 0$ it is sufficient to find a *barrier* $c \in \mathbb{R}$ such that $\Delta d_{\bar{E}} \leq c$ outside from E and $\Delta(-d_{X \setminus \bar{E}}) \geq c$ inside E .
- If no such *barrier* exists we find points $x \in X \setminus \bar{E}$ and $y \in E$, a regular *lower touching* function φ for $d_{\bar{E}}$ at x , an *upper touching* function ψ for $-d_{X \setminus \bar{E}}$ at y , such that

$$\Delta\varphi(x) > \Delta\psi(y).$$

- We play the same game as before simultaneously with φ and ψ . By continuity, there is a volume fixing perturbation with strictly smaller perimeter, contradiction.

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Thank you for your attention!