

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Boundary regularity and stability under lower Ricci bounds

Daniele Semola

Mathematical Institute, University of Oxford

Daniele.Semola@maths.ox.ac.uk

Curvature Constraints and Spaces of Metrics - Grenoble

Introduction

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth (K, μ) spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- optimal regularity of non smooth spaces.

Can we extend the quality of the effective boundary regularity to non smooth spaces?

Introduction

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\text{RCD}(K, N)$ spaces.

Introduction

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\text{RCD}(K, N)$ spaces.

Introduction

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\text{RCD}(K, N)$ spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\text{RCD}(K, N)$ spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\text{RCD}(K, N)$ spaces.

Question

Can we understand the regularity of **boundaries** under these assumptions?

Outline

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

- 1 Ricci curvature bounds
- 2 Effective interior regularity
- 3 Effective boundary regularity
- 4 Boundary regularity of non smooth spaces
- 5 Key Ideas

Ricci curvature bounds

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

• $\mathcal{M}_{N,K}$ is compactly generated and has finite asymptotic dimension

• $\mathcal{M}_{N,K}$ is uniformly locally contractible and has finite topological dimension

• $\mathcal{M}_{N,K}$ is uniformly locally \mathbb{R}^N -contractible and has finite topological dimension

• $\mathcal{M}_{N,K}$ is uniformly locally \mathbb{R}^N -contractible and has finite topological dimension

• $\mathcal{M}_{N,K}$ is uniformly locally \mathbb{R}^N -contractible and has finite topological dimension

• $\mathcal{M}_{N,K}$ is uniformly locally \mathbb{R}^N -contractible and has finite topological dimension

Ricci curvature bounds

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2 -Hessian bounds);

Ricci curvature bounds

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2 -Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '63];

Ricci curvature bounds

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2 -Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2 -Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2 -Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2 -Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

RCD spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, \mathfrak{m}) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds (asymptotically sharp);
- Cheeger-Croke on RCD(K, N) spaces; sharp
- Cheeger-Croke is proved under mollifying and some

• Cheeger-Croke on RCD(K, N) spaces

RCD spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '00], [Lot-Villani '09];

RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

RCD(K, N) metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are RCD(K, N);
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on RCD($0, N$) spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

Motivations

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

How do Riemannian manifolds in $\mathcal{M}_{n,K}$ look like?

How do $CD(n, K, \infty)$ spaces look like?

Motivations

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Motivations

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Motivations

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

Motivations

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

How do $\text{RCD}(K, N)$ spaces look like?

Motivations

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

How do $\text{RCD}(K, N)$ spaces look like?

Motivations

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

Question 3

How do $\text{RCD}(K, N)$ spaces look like?

Examples

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

There exist (M_n^c, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, \rho_n) \rightarrow (\mathbb{R}^2, d_{\text{Eucl}}, 0)$$

but

$$\text{inrad}(\rho_n) \rightarrow 0.$$

Let $\epsilon > 0$ be small and (M_n^c, g_n^c) with $K_n^c \geq 0$ and $\text{inrad}(\rho_n^c) \geq \epsilon$ and $(M_n, g_n, \rho_n) \rightarrow (M_n^c, g_n^c, \rho_n^c)$.

Examples

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{eucl}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$,
 $\text{vol}_n(B_1(p_n)) > \nu > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

Examples

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{eucl}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$,
 $\text{vol}_n(B_1(p_n)) > \nu > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

Examples

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{\text{eucl}}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y .

Examples

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{eucl}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y .

Examples

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{eucl}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y .

Examples

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \rightarrow (\mathbb{R}^2, d_{\text{eucl}}, 0)$$

but

$$\text{injrads}(p_n) \rightarrow 0.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\text{Ric}_n \geq -(n-1)$, $\text{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \rightarrow (Y, d_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y .

Volume, topology and ε -regularity, I

**Boundaries and
lower Ricci
bounds**

Daniele Semola

Ricci curvature
bounds

**Effective interior
regularity**

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Volume, topology and ε -regularity, I

Volume, topology and ε -regularity, I

Volume, topology and ε -regularity, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\text{Ric}_M \geq n - 1$;

ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(S^n)$;

then M is homeomorphic to S^n .

Volume, topology and ε -regularity, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\text{Ric}_M \geq n - 1$;

ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(S^n)$;

then M is homeomorphic to S^n .

Volume, topology and ε -regularity, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

- i) $\text{Ric}_M \geq n - 1$;
- ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(\mathbb{S}^n)$;

then M is homeomorphic to \mathbb{S}^n .

Volume, topology and ε -regularity, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

- i) $\text{Ric}_M \geq n - 1$;
- ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(\mathbb{S}^n)$;

then M is homeomorphic to \mathbb{S}^n .

Volume, topology and ε -regularity, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\text{Ric}_M \geq n - 1$;

ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(\mathbb{S}^n)$;

then M is homeomorphic to \mathbb{S}^n .

Volume, topology and ε -regularity, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

- i) $\text{Ric}_M \geq n - 1$;*
- ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(\mathbb{S}^n)$;*

then M is homeomorphic to \mathbb{S}^n .

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\text{Ric}_M \geq n - 1$,

$$\text{vol}(M) \sim \text{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0.$$

Volume, topology and ε -regularity, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

- i) $\text{Ric}_M \geq n - 1$;*
- ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(\mathbb{S}^n)$;*

then M is homeomorphic to \mathbb{S}^n .

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\text{Ric}_M \geq n - 1$,

$$\text{vol}(M) \sim \text{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0.$$

Volume, topology and ε -regularity, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\text{Ric}_M \geq n - 1$;

ii) $\text{vol}(M) \geq (1 - \varepsilon_n)\text{vol}(\mathbb{S}^n)$;

then M is homeomorphic to \mathbb{S}^n .

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\text{Ric}_M \geq n - 1$,

$$\text{vol}(M) \sim \text{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0.$$

Volume convergence

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

In general the volume is not continuous w.r.t. GH convergence.

The volume functional

$Vol : \mathcal{M} \rightarrow \mathbb{R}$

It is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of manifolds with bounded diameter and bounded Ricci curvature.

Volume convergence

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Remark

In general the volume is not continuous w.r.t. GH convergence.

The volume function

$$B_r(x) \mapsto \text{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{\alpha, -(n-1)}$.

Volume convergence

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Remark

In general the volume is not continuous w.r.t. GH convergence.

The volume function

$$B_r(x) \mapsto \text{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{\alpha, -(n-1)}$.

Volume convergence

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Remark

In general the volume is not continuous w.r.t. GH convergence.

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

The volume function

$$B_r(x) \mapsto \text{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{n, -(n-1)}$.

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas



Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\text{Ric}_M \geq 0$;

ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\text{Ric}_M \geq 0$;

ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\text{Ric}_M \geq 0$;

ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

- i) $\text{Ric}_M \geq 0$;
- ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

- i) $d_{\text{GH}}(B_r(q), B_r(0^n)) < \varepsilon$ for any $q \in M$ and $r > 0$.

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\text{Ric}_M \geq 0$;

ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

i) $\text{diam}(B_\varepsilon(q), B_\varepsilon(0^n)) < \varepsilon$ for any $q \in M$ and $\varepsilon > 0$;

ii) M^n is $C^{1-\beta}$ -biHolder homeomorphic to \mathbb{R}^n .

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\text{Ric}_M \geq 0$;

ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

a) $d_{GH}(B_r(q), B_r(0^n)) < \varepsilon r$ for any $q \in M$ and $r > 0$;

b) M^n is $C^{1-\varepsilon}$ -biHölder homeomorphic to \mathbb{R}^n .

Volume, topology and ε -regularity, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\text{Ric}_M \geq 0$;

ii) $\text{vol}(B_r(p)) \geq (1 - \delta)\text{vol}(B_r(0^n))$ for any $r > 0$;

then M^n is diffeomorphic to \mathbb{R}^n .

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

a) $d_{GH}(B_r(q), B_r(0^n)) < \varepsilon r$ for any $q \in M$ and $r > 0$;

b) M^n is $C^{1-\varepsilon}$ -biHölder homeomorphic to \mathbb{R}^n .

Including the boundary

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

The Euclidean half-space $(\mathbb{R}_+^n, d_{\text{eucl}}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds without boundary in $\mathcal{M}_{n-(n-1)}$.

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- a compact boundary (– nonnegative second fundamental form);
- a fixed uniformly bounded Ricci lower bound.

Including the boundary

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, J.D.G. '97)

*The Euclidean half-space $(\mathbb{R}_+^n, d_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** in $\mathcal{M}_{n, -(n-1)}$.*

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

boundary (Cheeger-Colding, J.D.G. '97)

Including the boundary

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, J.D.G. '97)

*The Euclidean half-space $(\mathbb{R}_+^n, d_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** in $\mathcal{M}_{n, -(n-1)}$.*

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

boundary (Cheeger-Colding, J.D.G. '97)

Including the boundary

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *J.D.G.* '97)

*The Euclidean half-space $(\mathbb{R}_+^n, d_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** in $\mathcal{M}_{n, -(n-1)}$.*

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

• convex boundary (= nonnegative second fundamental form);

Including the boundary

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *J.D.G.* '97)

*The Euclidean half-space $(\mathbb{R}_+^n, d_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** in $\mathcal{M}_{n, -(n-1)}$.*

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *J.D.G.* '97)

*The Euclidean half-space $(\mathbb{R}_+^n, d_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** in $\mathcal{M}_{n, -(n-1)}$.*

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- **convex** boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *J.D.G.* '97)

*The Euclidean half-space $(\mathbb{R}_+^n, d_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** in $\mathcal{M}_{n, -(n-1)}$.*

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- **convex** boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Boundary ε -regularity: topology

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{\text{GM}}(B_2(p), B_2^{\text{GM}}(0^n)) < \delta$, $\|g_M\| \geq 0$ and $\text{Ric}_M \geq -\delta$, then:

Boundary ε -regularity: topology

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta$, $H_{\partial M} \geq 0$ and $\text{Ric}_M \geq -\delta$, then:

- i) $d_{GH}(B_r(q), B_r^{\mathbb{R}^n_+}(0^n)) < \varepsilon r$, for any $0 < r < 1$ and any $q \in \partial M \cap B_1(p)$;
- ii) $B_1(p)$ is $C^{1-\varepsilon}$ -biHölder homeomorphic to $B_1(0^n) \subset \mathbb{R}^n_+$.

Boundary ε -regularity: topology

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}_+(0^n)) < \delta$, $H_{\partial M} \geq 0$ and $\text{Ric}_M \geq -\delta$, then:

i) $d_{GH}(B_r(q), B_r^{\mathbb{R}^n}_+(0^n)) < \varepsilon r$, for any $0 < r < 1$ and any $q \in \partial M \cap B_1(p)$;

ii) $B_1(p)$ is $C^{1-\varepsilon}$ -biHölder homeomorphic to $B_1(0^n) \subset \mathbb{R}^n_+$.

Boundary ε -regularity: topology

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta$, $H_{\partial M} \geq 0$ and $\text{Ric}_M \geq -\delta$, then:

- i) $d_{GH}(B_r(q), B_r^{\mathbb{R}^n_+}(0^n)) < \varepsilon r$, for any $0 < r < 1$ and any $q \in \partial M \cap B_1(p)$;
- ii) $B_1(p)$ is $C^{1-\varepsilon}$ -biHölder homeomorphic to $B_1(0^n) \subset \mathbb{R}_+^n$.

Boundary ε -regularity: volume

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$|K_M| \leq \delta \quad \text{and} \quad \text{Ric}_M \geq -\delta,$$

if $p \in M$ and

$$d_{\text{GH}}(B_2(p), B_2^{\text{sp}}(0^n)) < \delta,$$

then

Boundary ε -regularity: volume

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq -\delta,$$

if $p \in M$ and

$$d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0^n)) < \delta,$$

then

- i) $\partial M \cap B_2(p) \neq \emptyset$;
- ii) $(1 - \varepsilon)\omega_{n-1}r^{n-1} \leq \mathcal{H}^{n-1}(B_r(q) \cap \partial M) \leq (1 + \varepsilon)\omega_{n-1}r^{n-1}$,
for any $q \in \partial M \cap B_1(p)$ and any $0 < r < 1$.

Boundary ε -regularity: volume

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq -\delta,$$

if $p \in M$ and

$$d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0^n)) < \delta,$$

then

- i) $\partial M \cap B_2(p) \neq \emptyset$;
- ii) $(1 - \varepsilon)\omega_{n-1}r^{n-1} \leq \mathcal{H}^{n-1}(B_r(q) \cap \partial M) \leq (1 + \varepsilon)\omega_{n-1}r^{n-1}$,
for any $q \in \partial M \cap B_1(p)$ and any $0 < r < 1$.

Boundary ε -regularity: volume

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq -\delta,$$

if $p \in M$ and

$$d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0^n)) < \delta,$$

then

- i) $\partial M \cap B_2(p) \neq \emptyset$;
- ii) $(1 - \varepsilon)\omega_{n-1}r^{n-1} \leq \mathcal{H}^{n-1}(B_r(q) \cap \partial M) \leq (1 + \varepsilon)\omega_{n-1}r^{n-1}$,
for any $q \in \partial M \cap B_1(p)$ and any $0 < r < 1$.

Non collapsed spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Outline

This is inspired by the theory of noncollapsed Ricci limit spaces.

• Volume convergence holds in the strong

• Volume convergence holds in the strong

• Volume convergence holds in the strong

• Volume convergence holds in the strong

Non collapsed spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, \mathfrak{m}) where $\mathfrak{m} = \mathcal{H}^N$ are considered.

This is inspired by the theory of noncollapsed Ricci limit spaces.

Non collapsed spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- ★ Colding's volume convergence holds in this setting [De Philippis-Gigli '18]

Non collapsed spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];

Non collapsed spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε -regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε -regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε -regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\text{RCD}(K, N)$ spaces (X, d, m) where $m = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε -regularity theorem holds [Kapovitch-Mondino '21].

The boundary of a noncollapsed RCD space

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Let (M, μ) be a noncollapsed RCD(K, N) space with $K \geq 0$ and $N \geq 2$.

Let ∂M be the boundary of M with $\mu|_{\partial M} > 0$ and $\dim \partial M = N-1$.

Then (M, μ) is an RCD(K, N) space. Moreover,

There are only two possible tangent spaces on M :

• If ∂M is regular, the tangent is \mathbb{R}^N .

• If ∂M is not regular, the tangent is $\mathbb{R}^N \times \mathbb{R}^N$.

Let (M, μ) be a noncollapsed RCD(K, N) space with $K \geq 0$ and $N \geq 2$.

Then (M, μ) is an RCD(K, N) space. Moreover,

• If ∂M is regular,

The boundary of a noncollapsed RCD space

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth Riemannian manifold* with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

1. $(\mathbb{R}^n, d_{\text{eucl}}, \text{vol}_{\text{eucl}})$ (if M is smooth)

2. $(\mathbb{R}^n, d_{\text{eucl}}, \text{vol}_{\text{eucl}})$ (if M is not smooth)

The boundary of a noncollapsed RCD space

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth Riemannian manifold* with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

The boundary of a noncollapsed RCD space

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth Riemannian manifold* with

$$\|_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

• interior points are regular, the tangent is \mathbb{R}^n ;

• boundary points are regular, the tangent is \mathbb{R}^n .

The boundary of a noncollapsed RCD space

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth* Riemannian manifold with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

- interior points are regular, the tangent is \mathbb{R}^n ,
- boundary points have tangent *half spaces*, $\mathbb{R}^n_+ \subset \mathbb{R}^n$.

The boundary of a noncollapsed RCD space

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth Riemannian manifold* with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

- **interior points** are regular, the tangent is \mathbb{R}^n ;
- **boundary points** have tangent *half spaces*, $\mathbb{R}_+^n \subseteq \mathbb{R}^n$.

Definition

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ space. The **boundary** ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean **half-space** \mathbb{R}_+^N .

The boundary of a noncollapsed RCD space

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth Riemannian manifold* with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

- **interior points** are regular, the tangent is \mathbb{R}^n ;
- **boundary points** have tangent *half spaces*, $\mathbb{R}_+^n \subseteq \mathbb{R}^n$.

Definition

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ space. The **boundary** ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean **half-space** \mathbb{R}_+^N .

The boundary of a noncollapsed RCD space

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a *smooth Riemannian manifold* with

$$H_{\partial M} \geq 0 \quad \text{and} \quad \text{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\text{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M :

- **interior points** are regular, the tangent is \mathbb{R}^n ;
- **boundary points** have tangent *half spaces*, $\mathbb{R}_+^n \subseteq \mathbb{R}^n$.

Definition

Let (X, d, \mathcal{H}^N) be an $\text{RCD}(K, N)$ space. The **boundary** ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean **half-space** \mathbb{R}_+^N .

Stratification of singular sets

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$\mathcal{S}^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

$$\dim_{\mathcal{H}^k}(\mathcal{S}^k) \leq k.$$

What if $k = N$?

$$\dim_{\mathcal{H}^N}(\mathcal{S}^N) \leq N.$$

Stratification of singular sets

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$S^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

Stratification of singular sets

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Definition

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$\mathcal{S}^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

$$\dim_{\mathcal{H}^k}(\mathcal{S}^k) \leq k.$$

Stratification of singular sets

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Definition

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$\mathcal{S}^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

$$\dim_{\mathcal{H}^k}(\mathcal{S}^k) \leq k.$$

Stratification of singular sets

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Definition

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$\mathcal{S}^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.*'18)

$$\dim_H(\mathcal{S}^k) \leq k.$$

With this notation

$$\partial X = \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}.$$

Stratification of singular sets

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Definition

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$\mathcal{S}^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.*'18)

$$\dim_H(\mathcal{S}^k) \leq k.$$

With this notation

$$\partial X = \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}.$$

Stratification of singular sets

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Inspired by the stratification of singularities in GMT:

Definition

Given an $\text{RCD}(K, N)$ m.m.s. (X, d, \mathcal{H}^N) and $0 \leq k \leq N$, we set

$$\mathcal{S}^k := \{x \in X : \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.*'18)

$$\dim_H(\mathcal{S}^k) \leq k.$$

Remark

With this notation

$$\partial X = \overline{\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}}.$$

Boundary regularity for RCD spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(K, n)$ space. Then

Boundary regularity for RCD spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(K, n)$ space. Then

Boundary regularity for RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let (X, d, \mathcal{H}^n) be an RCD(K, n) space. Then

- ∂X is $(n - 1)$ -rectifiable with locally finite \mathcal{H}^{n-1} -measure;
- if $\mathbb{R}_+^n \in \text{Tan}_x(X, d, \mathcal{H}^n)$, then $\{\mathbb{R}_+^n\} = \text{Tan}_x(X, d, \mathcal{H}^n)$;
- ∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;
- X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.

Boundary regularity for RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(K, n)$ space. Then

- ∂X is $(n - 1)$ -rectifiable with locally finite \mathcal{H}^{n-1} -measure;
- if $\mathbb{R}_+^n \in \text{Tan}_x(X, d, \mathcal{H}^n)$, then $\{\mathbb{R}_+^n\} = \text{Tan}_x(X, d, \mathcal{H}^n)$;
- ∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;
- X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.

Boundary regularity for RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let (X, d, \mathcal{H}^n) be an RCD(K, n) space. Then

- ∂X is $(n - 1)$ -rectifiable with locally finite \mathcal{H}^{n-1} -measure;
- if $\mathbb{R}_+^n \in \text{Tan}_x(X, d, \mathcal{H}^n)$, then $\{\mathbb{R}_+^n\} = \text{Tan}_x(X, d, \mathcal{H}^n)$;
- ∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;
- X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.

Boundary regularity for RCD spaces

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let (X, d, \mathcal{H}^n) be an RCD(K, n) space. Then

- ∂X is $(n - 1)$ -rectifiable with locally finite \mathcal{H}^{n-1} -measure;
- if $\mathbb{R}_+^n \in \text{Tan}_x(X, d, \mathcal{H}^n)$, then $\{\mathbb{R}_+^n\} = \text{Tan}_x(X, d, \mathcal{H}^n)$;
- ∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;
- X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.

Boundary regularity for RCD spaces

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let (X, d, \mathcal{H}^n) be an $\text{RCD}(K, n)$ space. Then

- ∂X is $(n - 1)$ -rectifiable with locally finite \mathcal{H}^{n-1} -measure;
- if $\mathbb{R}_+^n \in \text{Tan}_x(X, d, \mathcal{H}^n)$, then $\{\mathbb{R}_+^n\} = \text{Tan}_x(X, d, \mathcal{H}^n)$;
- ∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;
- X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.

Boundary stability

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\partial X = \emptyset$.

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces with $\partial X_i \neq \emptyset$. Suppose that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\partial X = \emptyset$.

Boundary stability

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\partial X = \emptyset$.

Boundary stability

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\partial X = \emptyset$.

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\mathcal{H}^{n-1}(\partial X_i) \rightarrow \mathcal{H}^{n-1}(\partial X)$.

Boundary stability

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\partial X = \emptyset$.

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\mathcal{H}^{n-1}(\partial X_i) \rightarrow \mathcal{H}^{n-1}(\partial X)$.

Boundary stability

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\partial X = \emptyset$.

Theorem (Brué-Naber-S. '20)

Let $(X_i, d_i, \mathcal{H}^n)$ be $\text{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to (X, d, \mathcal{H}^n) . Then $\mathcal{H}^{n-1}(\partial X_i) \rightarrow \mathcal{H}^{n-1}(\partial X)$.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Most of these regularity results are peculiar of codimension one singularities.

- In [Otto-Shioya '94] an example of $2d$ Alexandrov space such that S^0 has locally infinite \mathcal{H}^0 measure is constructed.

- In [Otto-Shioya '94] an example of n -dim. Alexandrov space where S^{n-2} has nontrivial points is constructed.

- In [Otto-Shioya '94] an example of $2d$ noncollapsed Ricci limit space is built with a point $x \in S^0$ of which there are sequences of points $x_i \rightarrow x$ and $y_i \rightarrow x$ that do not split any line.

- In [Otto-Shioya '94] an example of $2d$ noncollapsed Ricci limit space is built with a point $x \in S^0$ of which there are sequences of points $x_i \rightarrow x$ and $y_i \rightarrow x$ that do not split any line.

- In [Otto-Shioya '94] an example of $2d$ noncollapsed Ricci limit space where S^0 is

- In [Otto-Shioya '94] an example of $2d$ noncollapsed Ricci limit space where S^0 is

Remarks

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Most of these regularity results are peculiar of codimension one singularities.

• in [Cian-Shioya '94] an example of $2d$ Alexandrov space such that S^0 has locally infinite \mathcal{H}^d measure is constructed.

• in [Li-Naber '20] an example of N -dim. Alexandrov space where S^{N-2} has no manifold points is constructed.

• in [Cian-Shioya '94] an example of $2d$ Alexandrov space such that S^0 has locally infinite \mathcal{H}^d measure is constructed.

• in [Li-Naber '20] an example of N -dim. Alexandrov space where S^{N-2} has no manifold points is constructed.

• in [Cian-Shioya '94] an example of $2d$ Alexandrov space such that S^0 has locally infinite \mathcal{H}^d measure is constructed.

• in [Li-Naber '20] an example of N -dim. Alexandrov space where S^{N-2} has no manifold points is constructed.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that \mathcal{S}^0 has locally infinite \mathcal{H}^0 measure is constructed.
- in [Li-Naber '20] an example of N -dim. Alexandrov space where \mathcal{S}^{N-2} has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If (X, d, \mathcal{H}^n) is a noncollapsed Ricci limit space, then \mathcal{S}^k is k -rectifiable for any $0 \leq k \leq n - 2$.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that \mathcal{S}^0 has locally infinite \mathcal{H}^0 measure is constructed.
- in [Li-Naber '20] an example of N -dim. Alexandrov space where \mathcal{S}^{N-2} has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If (X, d, \mathcal{H}^n) is a noncollapsed Ricci limit space, then \mathcal{S}^k is k -rectifiable for any $0 \leq k \leq n - 2$.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that \mathcal{S}^0 has locally infinite \mathcal{H}^0 measure is constructed.
- in [Li-Naber '20] an example of N -dim. Alexandrov space where \mathcal{S}^{N-2} has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If (X, d, \mathcal{H}^n) is a noncollapsed Ricci limit space, then \mathcal{S}^k is k -rectifiable for any $0 \leq k \leq n - 2$.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that \mathcal{S}^0 has locally infinite \mathcal{H}^0 measure is constructed.
- in [Li-Naber '20] an example of N -dim. Alexandrov space where \mathcal{S}^{N-2} has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If (X, d, \mathcal{H}^n) is a noncollapsed Ricci limit space, then \mathcal{S}^k is k -rectifiable for any $0 \leq k \leq n - 2$.

Connecting Analysis to Geometry, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Connecting Analysis to Geometry, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta;$

ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta;$

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

Connecting Analysis to Geometry, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta;$

ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta;$

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

Connecting Analysis to Geometry, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

aj) u has harmonic components;

Connecting Analysis to Geometry, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

- a) u has harmonic components;
- b) u is $(1+\varepsilon)$ -Lipschitz;

Connecting Analysis to Geometry, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;*
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;*

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

- a) u has harmonic components;*
- b) u is $(1 + \varepsilon)$ -Lipschitz;*
- c) $u : B_1(p) \rightarrow u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*
- d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p)$ away from a set U with $\text{vol}(B_1(p) \setminus U) \leq \varepsilon$;*
- e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

- a) *u has harmonic components;*
- b) *u is $(1 + \varepsilon)$ -Lipschitz;*
- c) *$u : B_1(p) \rightarrow u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*
- d) *u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p)$ away from a set U with $\text{vol}(B_1(p) \setminus U) \leq \varepsilon$;*
- e) *u is a $C^{1-\varepsilon}$ -biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

- a) u has harmonic components;
- b) u is $(1 + \varepsilon)$ -Lipschitz;
- c) $u : B_1(p) \rightarrow u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;
- d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p)$ away from a set U with $\text{vol}(B_1(p) \setminus U) \leq \varepsilon$;
- e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism.

Connecting Analysis to Geometry, I

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

- a) u has harmonic components;
- b) u is $(1 + \varepsilon)$ -Lipschitz;
- c) $u : B_1(p) \rightarrow u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;
- d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p)$ away from a set U with $\text{vol}(B_1(p) \setminus U) \leq \varepsilon$;
- e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism.

Connecting Analysis to Geometry, I

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\text{Ric}_M \geq -\delta$;
- ii) $d_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^n$ such that

- a) u has harmonic components;
- b) u is $(1 + \varepsilon)$ -Lipschitz;
- c) $u : B_1(p) \rightarrow u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;
- d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p)$ away from a set U with $\text{vol}(B_1(p) \setminus U) \leq \varepsilon$;
- e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism.

Harmonic splitting maps

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a harmonic δ -splitting map provided:

$\Delta u = 0$ in $B_{2r}(p)$ and $|\nabla u| \leq \delta$ in $B_{2r}(p)$.

$\Delta u = 0$ in $B_{2r}(p)$ and $|\nabla u| \leq \delta$ in $B_{2r}(p)$.

Harmonic splitting maps

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;

- $|\nabla u| \leq 1 + \delta$

-

$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}U\|^2 \leq \delta^2;$$

-

$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;

- $|\nabla u| \leq 1 + \delta$



$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}U\|^2 \leq \delta^2;$$



$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

Harmonic splitting maps

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$

- $$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$

- $$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

Harmonic splitting maps

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$

■

$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$

■

$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;

- $|\nabla u| \leq 1 + \delta$



$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$



$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$

■

$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$

■

$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- Integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;

- $|\nabla u| \leq 1 + \delta$



$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$



$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;

- $|\nabla u| \leq 1 + \delta$



$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$



$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;

- $|\nabla u| \leq 1 + \delta$



$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$



$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Definition

A map $u : B_{2r}(p) \rightarrow \mathbb{R}^k$ is a **harmonic δ -splitting map** provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$

■

$$\frac{r^2}{\text{vol}(B_r(p))} \int_{B_r(p)} \|\text{Hess}u\|^2 \leq \delta^2;$$

■

$$\frac{1}{\text{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Connecting Analysis to Geometry, II

**Boundaries and
lower Ricci
bounds**

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Connecting Analysis to Geometry, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $\| \cdot \|_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

Connecting Analysis to Geometry, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $H_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

Connecting Analysis to Geometry, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $H_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

Connecting Analysis to Geometry, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $H_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

all u has harmonic components,

Connecting Analysis to Geometry, II

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $H_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

Connecting Analysis to Geometry, II

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $\text{II}_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

a) u has harmonic components;

b) u is $(1 + \varepsilon)$ -Lipschitz;

c) $u : \partial M \cap B_1(p) \rightarrow u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;

d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;

e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.

Connecting Analysis to Geometry, II

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $\text{II}_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

a) u has harmonic components;

b) u is $(1 + \varepsilon)$ -Lipschitz;

c) $u : \partial M \cap B_1(p) \rightarrow u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;

d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;

e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.

Connecting Analysis to Geometry, II

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $H_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

a) u has harmonic components;

b) u is $(1 + \varepsilon)$ -Lipschitz;

c) $u : \partial M \cap B_1(p) \rightarrow u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;

d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;

e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.

Connecting Analysis to Geometry, II

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $H_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

a) u has harmonic components;

b) u is $(1 + \varepsilon)$ -Lipschitz;

c) $u : \partial M \cap B_1(p) \rightarrow u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;

d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;

e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.

Connecting Analysis to Geometry, II

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\text{Ric}_M \geq -\delta$ and $\text{II}_{\partial M} \geq 0$;

ii) $d_{GH}(B_2(p), B_2^{\mathbb{R}^n}(0)) < \delta$;

then there exists $u : B_1(p) \rightarrow \mathbb{R}^{n-1}$ such that

a) u has harmonic components;

b) u is $(1 + \varepsilon)$ -Lipschitz;

c) $u : \partial M \cap B_1(p) \rightarrow u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;

d) u is $(1 + \varepsilon)$ -biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;

e) u is a $C^{1-\varepsilon}$ -biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

- a maximal function argument is not sufficient here;
- the volume ratio

$$\frac{\text{vol}(B_r(x))}{\omega_n r^n}$$

is not uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to 1 if $\text{Ric} \geq 0$.

• The volume ratio is bounded away from 0 if $\text{Ric} \geq -\lambda$ and $\text{diam} \leq \frac{1}{\sqrt{\lambda}}$. This is a consequence of the Bishop-Gromov volume comparison theorem. In particular, if $\text{Ric} \geq -\lambda$ and $\text{diam} \leq \frac{1}{\sqrt{\lambda}}$ and $\text{vol}(B_r(x)) \leq \frac{1}{2} \omega_n r^n$ then $r \leq \frac{1}{\sqrt{\lambda}}$.

• The volume ratio is bounded away from 0 if $\text{Ric} \geq -\lambda$ and $\text{diam} \leq \frac{1}{\sqrt{\lambda}}$. This is a consequence of the Bishop-Gromov volume comparison theorem. In particular, if $\text{Ric} \geq -\lambda$ and $\text{diam} \leq \frac{1}{\sqrt{\lambda}}$ and $\text{vol}(B_r(x)) \leq \frac{1}{2} \omega_n r^n$ then $r \leq \frac{1}{\sqrt{\lambda}}$.

Remarks

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\text{vol}(B_r(x))}{\omega_n r^n}$$

is **not** uniformly pinched in general (while it is in the interior case);

Remarks

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\text{vol}(B_r(x))}{\omega_n r^n}$$

is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near $1/2$ if $x \in \partial M$.

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\text{vol}(B_r(x))}{\omega_n r^n}$$

is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Let $n \geq 1$ be fixed. For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, n) > 0$ such that if (M^n, g) is a smooth manifold with convex boundary and $\text{Ric}_M \geq -\varepsilon$, if $x \in \partial M$ and

$$\text{vol}(B_1(x)) \geq \frac{1}{2} \omega_n - \varepsilon,$$

then

$$d_{\text{GH}}(B_{1/2}(x), B_{1/2}^{\text{Eucl}}(0)) < \delta.$$

Remarks

Boundaries and lower Ricci bounds

Daniele Semola

Ricci curvature bounds

Effective interior regularity

Effective boundary regularity

Boundary regularity of non smooth spaces

Key Ideas

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\text{vol}(B_r(x))}{\omega_n r^n}$$

is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Let $n \geq 1$ be fixed. For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, n) > 0$ such that if (M^n, g) is a smooth manifold with convex boundary and $\text{Ric}_M \geq -\varepsilon$, if $x \in \partial M$ and

$$\text{vol}(B_1(x)) \geq \frac{1}{2} \omega_n - \varepsilon,$$

then

$$d_{\text{GH}}(B_{1/2}(x), B_{1/2}^{\text{Eucl}}(0)) < \delta.$$

Remarks

Boundaries and
lower Ricci
bounds

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\text{vol}(B_r(x))}{\omega_n r^n}$$

is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, n) > 0$ such that if (M^n, g) is a smooth manifold with convex boundary and $\text{Ric}_M \geq -\varepsilon$, if $x \in \partial M$ and

$$\text{vol}(B_1(x)) \geq \frac{1}{2}\omega_n - \varepsilon,$$

then

$$d_{GH}(B_{1/2}(x), B_{1/2}^{\mathbb{R}_+^n}(0)) < \delta.$$

**Boundaries and
lower Ricci
bounds**

Daniele Semola

Ricci curvature
bounds

Effective interior
regularity

Effective
boundary
regularity

Boundary
regularity of non
smooth spaces

Key Ideas

Thank you for your attention!