Boundary regularity and stability under lower Ricci bounds

Daniele Semola
Mathematical Institute, University of Oxford
Daniele.Semola@maths.ox.ac.uk

Curvature Constraints and Spaces of Metrics - Grenoble

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth RCD(K, N) spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\mathrm{RCD}(K, N)$ spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\mathrm{RCD}(K, N)$ spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\mathrm{RCD}(K,N)$ spaces.

Can we understand the regularity of boundaries under these assumptions?

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\mathrm{RCD}(K, N)$ spaces.

Question

Can we understand the regularity of boundaries under these assumptions?

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\mathrm{RCD}(K, N)$ spaces.

Question

Can we understand the regularity of boundaries under these assumptions?

Introduction

Lower Ricci curvature bounds, coupled with dimension upper bounds and lower volume bounds, force:

- effective regularity on smooth Riemannian manifolds;
- partial regularity of their limits;
- partial regularity of non smooth $\mathrm{RCD}(K, N)$ spaces.

Question

Can we understand the regularity of boundaries under these assumptions?

Outline

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

Ricci curvature bounds

We consider manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1, \infty)$.

Key properties on the background for $\mathcal{M}_{N,K}$:

- Bochner's inequality (many functions have effective L^2-Hessian bounds);
- Bishop-Gromov inequality [Bishop '63], [Gromov '82];
- splitting theorem [Cheeger-Gromoll '72];
- Gromov's precompactness theorem [Gromov '82] (we can consider limits in the (pm)GH topology).

RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

$\mathrm{RCD}(K,N)$ metric measure spaces (X,d,m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K,N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0,N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

RCD spaces

$\mathrm{RCD}(K, N)$ metric measure spaces (X, d, m) are non smooth spaces with Ricci bounded from below by $K \in \mathbb{R}$, dimension bounded above by $1 \leq N < \infty$ and looking Riemannian rather than Finsler.

- Ricci limits are $\mathrm{RCD}(K, N)$;
- Bishop-Gromov holds [Sturm '06], [Lott-Villani '09];
- splitting theorem on $\mathrm{RCD}(0, N)$ spaces [Gigli '13];
- the class RCD is *closed* under splittings and cone constructions [Gigli '13], [Ketterer '15];
- weak Bochner's inequality [Erbar-Kuwada-Sturm '15], [Ambrosio-Mondino-Savaré '15].

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

How do $\mathrm{RCD}(K,N)$ spaces look like?

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

How do $\mathrm{RCD}(K,N)$ spaces look like?

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

How do $\mathrm{RCD}(K,N)$ spaces look like?

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

How do $\mathrm{RCD}(K,N)$ spaces look like?

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

Question 3

How do $\mathrm{RCD}(K, N)$ spaces look like?

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

Question 3

How do $\mathrm{RCD}(K,N)$ spaces look like?

Motivations

Question 1

How do Riemannian manifolds in $\mathcal{M}_{N,K}$ look like?

Question 2

How do pmGH limits of manifolds in $\mathcal{M}_{N,K}$ look like? What happens to volume and topology under limits?

Question 3

How do $\mathrm{RCD}(K, N)$ spaces look like?

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathsf{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) \geq v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathsf{d}_Y, p)$ where

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M^2_n, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathrm{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '18]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathrm{d}_Y, y)$ where

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathrm{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '12]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathrm{d}_Y, y)$ where

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M^2_n, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathrm{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathrm{d}_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y.

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathrm{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathrm{d}_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y.

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathrm{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathrm{d}_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y.

Examples

Example [Naber-Zhang, *Geom. Topol.* '16]

There exist (M_n^2, g_n) with $K_n \geq 0$ such that

$$(M_n, g_n, p_n) \to (\mathbb{R}^2, \mathrm{d}_{eucl}, 0)$$

but

$$\mathrm{injrad}(p_n) \to 0\,.$$

Example [Colding-Naber, *Geom. Funct. Anal.* '13]

For any $n \geq 3$ there exist (M^n, g_n, p_n) with $\mathrm{Ric}_n \geq -(n-1)$, $\mathrm{vol}_n(B_1(p_n)) > v > 0$ and $(M_n, g_n, p_n) \to (Y, \mathrm{d}_Y, y)$ where

- all points of (Y, d_Y) have Euclidean tangent cone;
- d_Y is not induced by a C^β Riemannian metric for any $0 < \beta < 1$;
- angles are *not well defined* at y.

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '96)

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0.$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '96)

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0 .$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '95)

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0 .$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '96)

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0 .$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\mathrm{Ric}_M \geq n - 1$,

$$\mathrm{vol}(M) \to \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \to 0\,.$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\mathrm{Ric}_M \geq n - 1$,

$$\mathrm{vol}(M) \to \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \to 0.$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n-1$;

ii) $\mathrm{vol}(M) \geq (1-\varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\mathrm{Ric}_M \geq n-1$,

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0 \,.$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\mathrm{Ric}_M \geq n - 1$,

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0\,.$$

Volume, topology and ε-regularity, I

Theorem (Perelman, *J. Amer. Math. Soc.* '94)

For any $n \geq 2$ there exists $\varepsilon_n > 0$ s.t. if (M^n, g) satisfies

i) $\mathrm{Ric}_M \geq n - 1$;

ii) $\mathrm{vol}(M) \geq (1 - \varepsilon_n)\mathrm{vol}(\mathbb{S}^n)$;

then M is homeomorphic to $\mathbb{S}^n$.

Theorem (Colding, *Invent. Math.* '95)

For any $n \geq 2$ and any (M^n, g) such that $\mathrm{Ric}_M \geq n - 1$,

$$\mathrm{vol}(M) \sim \mathrm{vol}(\mathbb{S}^n)$$

if and only if

$$d_{GH}(M, \mathbb{S}^n) \sim 0\,.$$

Volume convergence

Remark

In general the volume is not continuous w.r.t. GH convergence.

Theorem (Colding, Ann. Math. '97, Cheeger-Colding, JDG '97)

The volume function

$$B_r(x) \mapsto \mathrm{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{n,-(n-1)}$.

Volume convergence

Remark

In general the volume is not continuous w.r.t. GH convergence.

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

The volume function

$$B_r(x) \mapsto \mathrm{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{n,-(n-1)}$.

Volume convergence

Remark

In general the volume is not continuous w.r.t. GH convergence.

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

The volume function

$$B_r(x) \mapsto \mathrm{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{n,-(n-1)}$.

Volume convergence

Remark

In general the volume is not continuous w.r.t. GH convergence.

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

The volume function

$$B_r(x) \mapsto \mathrm{vol}(B_r(x))$$

is continuous w.r.t. the pointed Gromov-Hausdorff topology on the space of metric balls inside complete manifolds in $\mathcal{M}_{n,-(n-1)}$.

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1 - \delta)\mathrm{vol}(B_r(0^n))$ *for any $r > 0$;*

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1-\delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1-\delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1-\delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1-\delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

a) $d_{GH}(B_r(q), B_r(0^n)) < \varepsilon r$ *for any* $q \in M$ *and* $r > 0$;

b) *M^n is $C^{1-\varepsilon}$-biHölder homeomorphic to $\mathbb{R}^n$.*

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1-\delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

a) $d_{GH}(B_r(q), B_r(0^n)) < \varepsilon r$ *for any* $q \in M$ *and* $r > 0$;

b) *M^n is $C^{1-\varepsilon}$-biHölder homeomorphic to $\mathbb{R}^n$.*

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1 - \delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.
If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

a) $d_{GH}(B_r(q), B_r(0^n)) < \varepsilon r$ *for any $q \in M$ and $r > 0$;*

b) *M^n is $C^{1-\varepsilon}$-biHölder homeomorphic to $\mathbb{R}^n$.*

Volume, topology and ε-regularity, II

Theorem (Colding, *Ann. Math.* '97, Cheeger-Colding, *J.D.G.* '97)

There exists $\delta = \delta(n) > 0$ such that if (M^n, g) verifies

i) $\mathrm{Ric}_M \geq 0$;

ii) $\mathrm{vol}(B_r(p)) \geq (1-\delta)\mathrm{vol}(B_r(0^n))$ *for any* $r > 0$;

then M^n is diffeomorphic to $\mathbb{R}^n$.

If $\varepsilon > 0$ and $\delta < \delta(\varepsilon, N)$ then:

a) $d_{GH}(B_r(q), B_r(0^n)) < \varepsilon r$ *for any* $q \in M$ *and* $r > 0$;

b) *M^n is $C^{1-\varepsilon}$-biHölder homeomorphic to $\mathbb{R}^n$.*

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathsf{d}_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** *in $\mathcal{M}_{n,-(n-1)}$.*

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathrm{d}_{eucl}, \mathcal{H}^n)$ *is not a non collapsed pGH limit of a sequence of manifolds* **without boundary** *in* $\mathcal{M}_{n,-(n-1)}$.

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathrm{d}_{eucl}, \mathcal{H}^n)$ *is not a non collapsed pGH limit of a sequence of manifolds* **without boundary** *in* $\mathcal{M}_{n,-(n-1)}$.

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathrm{d}_{eucl}, \mathcal{H}^n)$ *is not a non collapsed pGH limit of a sequence of manifolds* **without boundary** *in* $\mathcal{M}_{n,-(n-1)}$.

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathrm{d}_{eucl}, \mathcal{H}^n)$ *is not a non collapsed pGH limit of a sequence of manifolds* **without boundary** *in* $\mathcal{M}_{n,-(n-1)}$.

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathrm{d}_{eucl}, \mathcal{H}^n)$ is not a non collapsed pGH limit of a sequence of manifolds **without boundary** *in $\mathcal{M}_{n,-(n-1)}$.*

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary ($=$ nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Including the boundary

Theorem (Cheeger-Colding, *J.D.G.* '97)

The Euclidean half-space $(\mathbb{R}^n_+, \mathrm{d}_{eucl}, \mathcal{H}^n)$ *is not a non collapsed pGH limit of a sequence of manifolds* **without boundary** *in* $\mathcal{M}_{n,-(n-1)}$.

Question

What happens if we include boundaries in the theory?

We first focus on smooth manifolds with

- convex boundary (= nonnegative second fundamental form);
- Ricci uniformly bounded from below in the interior.

Boundary ε-regularity: topology

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta$, $II_{\partial M} \geq 0$ and $\mathrm{Ric}_M \geq -\delta$, then:

Boundary ε-regularity: topology

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta$, $II_{\partial M} \geq 0$ and $\mathrm{Ric}_M \geq -\delta$, then:

i) *$d_{GH}(B_r(q), B_r^{\mathbb{R}^n_+}(0^n)) < \varepsilon r$, for any $0 < r < 1$ and any $q \in \partial M \cap B_1(p)$;*

ii) *$B_1(p)$ is $C^{1-\varepsilon}$-biHölder homeomorphic to $B_1(0^n) \subset \mathbb{R}^n_+$.*

Boundary ε-regularity: topology

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta$, $II_{\partial M} \geq 0$ and $\mathrm{Ric}_M \geq -\delta$, then:

i) *$d_{GH}(B_r(q), B_r^{\mathbb{R}^n_+}(0^n)) < \varepsilon r$, for any $0 < r < 1$ and any $q \in \partial M \cap B_1(p)$;*

ii) *$B_1(p)$ is $C^{1-\varepsilon}$-biHölder homeomorphic to $B_1(0^n) \subset \mathbb{R}^n_+$.*

Boundary ε-regularity: topology

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that if (M^n, g) is a smooth Riemannian manifold with boundary, $p \in M$, $d_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta$, $II_{\partial M} \geq 0$ and $\mathrm{Ric}_M \geq -\delta$, then:

i) $d_{GH}(B_r(q), B_r^{\mathbb{R}^n_+}(0^n)) < \varepsilon r$, *for any* $0 < r < 1$ *and any* $q \in \partial M \cap B_1(p)$;

ii) $B_1(p)$ *is* $C^{1-\varepsilon}$*-biHölder homeomorphic to* $B_1(0^n) \subset \mathbb{R}^n_+$.

Boundary ε-regularity: volume

Theorem (Bruè-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq -\delta\,,$$

if $p \in M$ and

$$\mathsf{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta\,,$$

then

Boundary ε-regularity: volume

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq -\delta\,,$$

if $p \in M$ and

$$\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta\,,$$

then

i) $\partial M \cap B_2(p) \neq \emptyset$;

ii) $(1-\varepsilon)\omega_{n-1} r^{n-1} \leq \mathcal{H}^{n-1}(B_r(q) \cap \partial M) \leq (1+\varepsilon)\omega_{n-1} r^{n-1}$, *for any $q \in \partial M \cap B_1(p)$ and any $0 < r < 1$.*

Boundary ε-regularity: volume

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq -\delta\,,$$

if $p \in M$ and

$$\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta\,,$$

then

i) $\partial M \cap B_2(p) \neq \emptyset$;

ii) $(1-\varepsilon)\omega_{n-1} r^{n-1} \leq \mathcal{H}^{n-1}(B_r(q) \cap \partial M) \leq (1+\varepsilon)\omega_{n-1} r^{n-1}$, *for any $q \in \partial M \cap B_1(p)$ and any $0 < r < 1$.*

Boundary ε-regularity: volume

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $0 < \varepsilon < \varepsilon(n)$ there exists $\delta = \delta(\varepsilon, n)$ such that for any (M^n, g) with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq -\delta \,,$$

if $p \in M$ and

$$\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0^n)) < \delta \,,$$

then

i) $\partial M \cap B_2(p) \neq \emptyset$;
ii) $(1-\varepsilon)\omega_{n-1} r^{n-1} \leq \mathcal{H}^{n-1}(B_r(q) \cap \partial M) \leq (1+\varepsilon)\omega_{n-1} r^{n-1} \,,$ *for any $q \in \partial M \cap B_1(p)$ and any $0 < r < 1$.*

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K,N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K, N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K,N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K,N)$ spaces (X,d,m) where $\mathrm{m}=\mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K, N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K, N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K,N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

Non collapsed spaces

In [De Philippis-Gigli '18], [Kitabeppu '17] and [Kapovitch-Mondino '21] $\mathrm{RCD}(K, N)$ spaces (X, d, m) where $\mathrm{m} = \mathcal{H}^N$ are considered.

Remark

This is inspired by the theory of noncollapsed Ricci limit spaces.

- Colding's volume convergence holds in this setting [De Philippis-Gigli '18];
- tangent cones are metric cones [De Philippis-Gigli '18];
- volume rigidity and almost volume rigidity hold [De Philippis-Gigli '16-'18];
- the topological ε-regularity theorem holds [Kapovitch-Mondino '21].

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$\mathrm{II}_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq K.$$

Then (M, d_g, vol) is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half spaces*, $\mathbb{R}^n_+ \subset \mathbb{R}^n$.

Definition

Let $(X, d, \mathscr{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half* spaces, $\mathbb{R}^n_+ \subset \mathbb{R}^n$.

Definition

Let $(X, \mathsf{d}, \mathscr{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half* spaces, $\mathbb{R}^n_+ \subset \mathbb{R}^n$.

Definition

Let $(X, \mathsf{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half* spaces, $\mathbb{R}^n_+ \subset \mathbb{R}^n$.

Definition

Let $(X, \mathsf{d}, \mathscr{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half spaces*, $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$.

Definition

Let $(X, \mathrm{d}, \mathscr{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad \text{and} \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half spaces*, $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$.

Definition

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad and \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half spaces*, $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$.

Definition

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

The boundary of a noncollapsed RCD space

Theorem (Han, *Adv. Math.* '20)

Let (M^n, g) be a smooth Riemannian manifold with

$$II_{\partial M} \geq 0 \quad \textit{and} \quad \mathrm{Ric}_M \geq K\,.$$

Then $(M, \mathrm{d}_g, \mathrm{vol})$ is an $\mathrm{RCD}(K, n)$ metric measure space.

There are only two possible tangent spaces on M:

- interior points are regular, the tangent is $\mathbb{R}^n$;
- boundary points have tangent *half spaces*, $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$.

Definition

Let $(X, \mathrm{d}, \mathcal{H}^N)$ be an $\mathrm{RCD}(K, N)$ space. The boundary ∂X of X is the topological closure of the set of points where a tangent cone is a Euclidean half-space $\mathbb{R}^N_+$.

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Definition

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,d,\mathcal{H}^N)$ and $0\le k\le N$, we set

$$\mathcal{S}^k := \{x\in X\ :\ \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}\,.$$

Theorem (De Philippis-Gigli [illegible])

$$\dim_H(\mathcal{S}^k)\le k\,.$$

With this notation

$$\partial X = \overline{\mathcal{S}^{N-1}\setminus\mathcal{S}^{N-2}}\,.$$

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Definition

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,d,\mathcal{H}^N)$ and $0\le k\le N$, we set

$$\mathcal{S}^k := \{x\in X\ :\ \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}\,.$$

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Definition

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X, d, \mathcal{H}^N)$ and $0 \le k \le N$, we set

$$\mathcal{S}^k := \{x \in X \,:\, \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}\,.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.* '18)

$$\dim_H(\mathcal{S}^k) \le k\,.$$

With this notation

$$\partial X = \mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}\,.$$

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Definition

Given an $\mathrm{RCD}(K, N)$ m.m.s. $(X, d, \mathcal{H}^N)$ and $0 \le k \le N$, we set

$$\mathcal{S}^k := \{x \in X \,:\, \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}\,.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.*'18)

$$\dim_H(\mathcal{S}^k) \le k\,.$$

Remark

With this notation

$$\partial X = \overline{\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}}\,.$$

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Definition

Given an $\mathrm{RCD}(K, N)$ m.m.s. $(X, d, \mathcal{H}^N)$ and $0 \le k \le N$, we set

$$\mathcal{S}^k := \{x \in X \,:\, \text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}\,.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.*'18)

$$\dim_H(\mathcal{S}^k) \le k\,.$$

Remark

With this notation

$$\partial X = \overline{\mathcal{S}^{N-1} \setminus \mathcal{S}^{N-2}}\,.$$

Stratification of singular sets

Inspired by the stratification of singularities in GMT:

Definition

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,d,\mathcal{H}^N)$ and $0\le k\le N$, we set

$$\mathcal{S}^k := \{x\in X\,:\,\text{no tangent at } x \text{ splits } \mathbb{R}^{k+1}\}\,.$$

Theorem (De Philippis-Gigli, *J. Éc. polytech. Math.*'18)

$$\dim_H(\mathcal{S}^k)\le k\,.$$

Remark

With this notation

$$\partial X = \overline{\mathcal{S}^{N-1}\setminus\mathcal{S}^{N-2}}\,.$$

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Bruè-Naber-S. '20)

Let $(X, \mathsf{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Bruè-Naber-S. '20)

Let $(X, \mathsf{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

- *∂X is $(n-1)$-rectifiable with locally finite $\mathcal{H}^{n-1}$-measure;*
- *if $\mathbb{R}^n_+ \in \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$, then $\{\mathbb{R}^n_+\} = \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$;*
- *∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;*
- *X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.*

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

- *∂X is $(n-1)$-rectifiable with locally finite $\mathcal{H}^{n-1}$-measure;*
- *if $\mathbb{R}^n_+ \in \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$, then $\{\mathbb{R}^n_+\} = \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$;*
- *∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;*
- *X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.*

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

- *∂X is $(n-1)$-rectifiable with locally finite $\mathcal{H}^{n-1}$-measure;*
- *if $\mathbb{R}^n_+ \in \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$, then $\{\mathbb{R}^n_+\} = \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$;*
- *∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;*
- *X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.*

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

- ∂X *is* $(n-1)$*-rectifiable with locally finite* $\mathcal{H}^{n-1}$*-measure;*
- *if* $\mathbb{R}^n_+ \in \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$*, then* $\{\mathbb{R}^n_+\} = \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$*;*
- ∂X *is biHölder homeomorphic to a smooth manifold away from a codimension* 2 *set;*
- *X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.*

Boundary regularity for RCD spaces

Rectifiability and measure bounds for ∂X were conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X, \mathrm{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space. Then

- *∂X is $(n-1)$-rectifiable with locally finite $\mathcal{H}^{n-1}$-measure;*
- *if $\mathbb{R}^n_+ \in \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$, then $\{\mathbb{R}^n_+\} = \mathrm{Tan}_x(X, \mathrm{d}, \mathcal{H}^n)$;*
- *∂X is biHölder homeomorphic to a smooth manifold away from a codimension 2 set;*
- *X is biHölder homeomorphic to a smooth manifold with boundary away from a codimension 2 set.*

Boundary stability

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Bruè-Naber-S. '20)

Let $(X_i, \mathsf{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to $(X, \mathsf{d}, \mathcal{H}^n)$. Then $\partial X = \emptyset$.

Theorem (Bruè-Naber-S. '20)

Let $(X_i, \mathsf{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to $(X, \mathsf{d}, \mathcal{H}^n)$. Then $\mathcal{H}^{n-1}(\partial X_i) \to \mathcal{H}^{n-1}(\partial X)$.

Boundary stability

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Bruè-Naber-S. '20)

Let $(X_i, \mathsf{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to $(X, \mathsf{d}, \mathcal{H}^n)$. Then $\partial X = \emptyset$.

Theorem (Bruè-Naber-S. '20)

Let $(X_i, \mathsf{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to $(X, \mathsf{d}, \mathcal{H}^n)$. Then $\mathcal{H}^{n-1}(\partial X_i) \to \mathcal{H}^{n-1}(\partial X)$.

Boundary stability

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to $(X, \mathrm{d}, \mathcal{H}^n)$. Then $\partial X = \emptyset$.

Theorem (Brué-Naber-S. '20)

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to $(X, \mathrm{d}, \mathcal{H}^n)$. Then $\mathcal{H}^{n-1}(\partial X_i) \to \mathcal{H}^{n-1}(\partial X)$.

Boundary stability

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces. Suppose that $\partial X_i = \emptyset$ and that they converge in the GH topology to $(X, \mathrm{d}, \mathcal{H}^n)$. Then $\partial X = \emptyset$.

Theorem (Brué-Naber-S. '20)

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ be $\mathrm{RCD}(K, n)$ metric measure spaces with diameter bounded above by $D > 0$. Suppose that they converge in the GH topology to $(X, \mathrm{d}, \mathcal{H}^n)$. Then $\mathcal{H}^{n-1}(\partial X_i) \to \mathcal{H}^{n-1}(\partial X)$.

Boundary stability

Stability of spaces without boundary was conjectured in [De Philippis-Gigli '18] and [Kapovitch-Mondino '21].

Theorem (Brué-Naber-S. '20)

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ *be* $\mathrm{RCD}(K, n)$ *metric measure spaces. Suppose that* $\partial X_i = \emptyset$ *and that they converge in the GH topology to* $(X, \mathrm{d}, \mathcal{H}^n)$*. Then* $\partial X = \emptyset$*.*

Theorem (Brué-Naber-S. '20)

Let $(X_i, \mathrm{d}_i, \mathcal{H}^n)$ *be* $\mathrm{RCD}(K, n)$ *metric measure spaces with diameter bounded above by* $D > 0$*. Suppose that they converge in the GH topology to* $(X, \mathrm{d}, \mathcal{H}^n)$*. Then* $\mathcal{H}^{n-1}(\partial X_i) \to \mathcal{H}^{n-1}(\partial X)$*.*

Remarks

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that $\mathcal{S}^0$ has locally infinite $\mathcal{H}^0$ measure is constructed.
- in [Li-Naber '20] an example of N-dim. Alexandrov space where $\mathcal{S}^{N-2}$ has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, Ann. Math. '21)

If $(X, \mathsf{d}, \mathcal{H}^n)$ is a noncollapsed Ricci limit space, then $\mathcal{S}^k$ is k-rectifiable for any $0 \leq k \leq n-2$.

Remarks

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that $\mathcal{S}^0$ has locally infinite $\mathcal{H}^0$ measure is constructed.
- in [Li-Naber '20] an example of N-dim. Alexandrov space where $\mathcal{S}^{N-2}$ has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, Ann. Math. '21)

If $(X, d, \mathcal{H}^n)$ is a noncollapsed Ricci limit space, then $\mathcal{S}^k$ is k-rectifiable for any $0 \le k \le n-2$.

Remarks

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of 2*d* Alexandrov space such that $\mathcal{S}^0$ has locally infinite $\mathcal{H}^0$ measure is constructed.
- in [Li-Naber '20] an example of *N*-dim. Alexandrov space where $\mathcal{S}^{N-2}$ has no manifold points is constructed.
- in [Colding-Naber '13] an example of 3*d* noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If $(X, \mathrm{d}, \mathcal{H}^n)$ is a noncollapsed Ricci limit space, then $\mathcal{S}^k$ is k-rectifiable for any $0 \leq k \leq n-2$.

Remarks

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of 2d Alexandrov space such that $\mathcal{S}^0$ has locally infinite $\mathcal{H}^0$ measure is constructed.
- in [Li-Naber '20] an example of N-dim. Alexandrov space where $\mathcal{S}^{N-2}$ has no manifold points is constructed.
- in [Colding-Naber '13] an example of 3d noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If $(X, \mathrm{d}, \mathcal{H}^n)$ is a noncollapsed Ricci limit space, then $\mathcal{S}^k$ is k-rectifiable for any $0 \le k \le n-2$.

Remarks

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of $2d$ Alexandrov space such that $\mathcal{S}^0$ has locally infinite $\mathcal{H}^0$ measure is constructed.
- in [Li-Naber '20] an example of N-dim. Alexandrov space where $\mathcal{S}^{N-2}$ has no manifold points is constructed.
- in [Colding-Naber '13] an example of $3d$ noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If $(X, \mathrm{d}, \mathcal{H}^n)$ is a noncollapsed Ricci limit space, then $\mathcal{S}^k$ is k-rectifiable for any $0 \leq k \leq n-2$.

Remarks

Most of these regularity results are peculiar of codimension one singularities.

- in [Otsu-Shioya '94] an example of 2d Alexandrov space such that $\mathcal{S}^0$ has locally infinite $\mathcal{H}^0$ measure is constructed.
- in [Li-Naber '20] an example of N-dim. Alexandrov space where $\mathcal{S}^{N-2}$ has no manifold points is constructed.
- in [Colding-Naber '13] an example of 3d noncollapsed Ricci limit space is build with a point $x \in \mathcal{S}^1 \setminus \mathcal{S}^0$ where there are tangents splitting a line and tangents that do not split any line.

Theorem (Cheeger-Jiang-Naber, *Ann. Math.* '21)

If $(X, \mathrm{d}, \mathcal{H}^n)$ is a noncollapsed Ricci limit space, then $\mathcal{S}^k$ is k-rectifiable for any $0 \le k \le n-2$.

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathsf{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Connecting Analysis to Geometry, I

Theorem (Cheeger-Colding, *Ann. Math.* '96, Cheeger-Naber, *Ann. Math.* '15, Cheeger-Jiang-Naber, *Ann. Math.* '21)

Let (M^n, g) be a smooth Riemannian manifold. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2(0^n)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^n$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : B_1(p) \to u(B_1(p)) \subset \mathbb{R}^n$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p)$ away from a set U with $\mathrm{vol}(B_1(p) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism.*

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess} u\|^2 \leq \delta^2 ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \leq \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \le 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \le \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \le \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \leq \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \leq \delta^2 \,;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \,.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \leq \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \leq \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess} u\|^2 \leq \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess}\, u\|^2 \leq \delta^2 \,;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \,.$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Harmonic splitting maps

Definition

A map $u : B_{2r}(p) \to \mathbb{R}^k$ is a harmonic δ-splitting map provided:

- it has harmonic components;
- $|\nabla u| \leq 1 + \delta$
- $$\frac{r^2}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} \|\mathrm{Hess} u\|^2 \leq \delta^2 \, ;$$
- $$\frac{1}{\mathrm{vol}(B_r(p))} \int_{B_r(p)} |\nabla u_i \cdot \nabla u_j - \delta_{ij}|^2 \leq \delta^2 \, .$$

- Integral bounds turn into pointwise bounds away from small sets via maximal function arguments;
- integral bounds propagate through scales up to controlled errors.

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ *such that*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ such that

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$*;*

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$*;*

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ *such that*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$*;*

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$*;*

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ such that

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ *such that*

a) *u has harmonic components;*

b) *u is* $(1+\varepsilon)$*-Lipschitz;*

c) $u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ *is a smooth diffeomorphism;*

d) *u is* $(1+\varepsilon)$*-biLipschitz on* $B_1(p) \cap \partial M$ *away from a set U with* $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) < \varepsilon$;

e) *u is a* $C^{1-\varepsilon}$*-biHölder homeomorphism between* $\partial M \cap B_1(p)$ *and its image.*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ *such that*

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. ’20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;
- ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ such that

- a) *u has harmonic components;*
- b) *u is $(1+\varepsilon)$-Lipschitz;*
- c) *$u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;*
- d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;*
- e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

- i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;
- ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ such that

- a) *u has harmonic components;*
- b) *u is $(1+\varepsilon)$-Lipschitz;*
- c) *$u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;*
- d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;*
- e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$*;*

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$*;*

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ such that

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) *$u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}\left((\partial M \cap B_1(p)) \setminus U\right) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.*

Connecting Analysis to Geometry, II

Theorem (Brué-Naber-S. '20)

Let (M^n, g) be a smooth Riemannian manifold with boundary. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, n) > 0$ such that if

i) $\mathrm{Ric}_M \geq -\delta$ *and* $II_{\partial M} \geq 0$;

ii) $\mathrm{d}_{GH}(B_2(p), B_2^{\mathbb{R}^n_+}(0)) < \delta$;

then there exists $u : B_1(p) \to \mathbb{R}^{n-1}$ *such that*

a) *u has harmonic components;*

b) *u is $(1+\varepsilon)$-Lipschitz;*

c) $u : \partial M \cap B_1(p) \to u(\partial M \cap B_1(p)) \subset \mathbb{R}^{n-1}$ *is a smooth diffeomorphism;*

d) *u is $(1+\varepsilon)$-biLipschitz on $B_1(p) \cap \partial M$ away from a set U with $\mathcal{H}^{n-1}((\partial M \cap B_1(p)) \setminus U) \leq \varepsilon$;*

e) *u is a $C^{1-\varepsilon}$-biHölder homeomorphism between $\partial M \cap B_1(p)$ and its image.*

Remarks

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\mathrm{vol}(B_r(x))}{\omega_n r^n}$$

is **not** uniformly pinched in general (while it is in the interior case);

Remarks

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\mathrm{vol}(B_r(x))}{\omega_n r^n}$$

  is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$

Remarks

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\mathrm{vol}(B_r(x))}{\omega_n r^n}$$

  is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Remarks

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\mathrm{vol}(B_r(x))}{\omega_n r^n}$$

  is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Theorem (Bruè-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, n) > 0$ such that if (M^n, g) is a smooth manifold with convex boundary and $\mathrm{Ric}_M \geq -\varepsilon$, if $x \in \partial M$ and

$$\mathrm{vol}(B_1(x)) \geq \frac{1}{2}\omega_n - \varepsilon ,$$

then

$$\mathrm{d}_{GH}(B_{1/2}(x), B^{\mathbb{R}^n_+}_{1/2}(0)) < \delta .$$

Remarks

- a maximal function argument is **not** sufficient here;
- the volume ratio

$$\frac{\mathrm{vol}(B_r(x))}{\omega_n r^n}$$

  is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Remarks

- a maximal function argument is **not** sufficient here;
- the volume ratio
$$\frac{\mathrm{vol}(B_r(x))}{\omega_n r^n}$$
is **not** uniformly pinched in general (while it is in the interior case); it is uniformly pinched near to $1/2$ if $x \in \partial M$.

Theorem (Brué-Naber-S. '20)

Let $n \geq 1$ be fixed. For any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta, n) > 0$ such that if (M^n, g) is a smooth manifold with convex boundary and $\mathrm{Ric}_M \geq -\varepsilon$, if $x \in \partial M$ and

$$\mathrm{vol}(B_1(x)) \geq \frac{1}{2}\omega_n - \varepsilon\,,$$

then

$$\mathrm{d}_{GH}(B_{1/2}(x), B^{\mathbb{R}^n_+}_{1/2}(0)) < \delta\,.$$

Thank you for your attention!