$$\delta_1\delta_2(a\cdot b) = \delta_1[\delta_2(a)b + a\,\delta_2(b)]$$
$$= \delta_1\delta_2(a)b + \underbrace{\delta_2(a)\,\delta_1(b)} + \underbrace{\delta_1(a)\delta_2(b)}$$
$$+\ a\,\delta_1\delta_2(b)$$

$$\delta_2\delta_1(ab) = \delta_2[\delta_1(a)b + a\,\delta_1(b)]$$
$$= \delta_2\delta_1(a)b + \underbrace{\delta_1(a)\,\delta_2(b)} + \underbrace{\delta_2(a)\delta_1(b)}$$
$$+\ a\cdot\delta_2\delta_1(b).$$

Hence.

$$(\delta_1\delta_2 - \delta_2\delta_1)(a\cdot b) = (\delta_1\delta_2 - \delta_2\delta_1)(a)\cdot b$$
$$+\ a(\delta_1\delta_2 - \delta_2\delta_1)(b)$$

$\square$

We can apply this to $\mathrm{Vect}^\infty(M)$: given $X, Y \in \mathrm{Vect}^\infty(M)$ we conclude from Lemma 3.22 that $\alpha X\cdot\alpha Y - \alpha Y\cdot\alpha X \in \mathrm{Der}(C^\infty(M))$ and hence by Prop. 3.20 it corresponds to an element of $\mathrm{Vect}^\infty(M)$.

Definition 3.23 [Bracket of vector fields]
The bracket $[X,Y]$ of two vector fields $X, Y \in \mathrm{Vect}^\infty(M)$ is the unique element in $\mathrm{Vect}^\infty(M)$ such that

$$\alpha([X,Y]) = \alpha X\cdot\alpha Y - \alpha Y\cdot\alpha X.$$

More generally we can formalize this operation as the following:

Definition 3.24 [Bracket of endomorphisms]
If V is any $\mathbb{K}$-vector space, the bracket: $[T_1, T_2] \in End(V)$ of two endomorphisms T_1, T_2 is
$$[T_1, T_2] := T_1 T_2 - T_2 T_1$$

If A is a $\mathbb{K}$-algebra the bracket operation is a bilinear map on $End(A)$ preserving $Der(A)$.

End of lect 27/03

The map
$$End(V) \times End(V) \longrightarrow End(V)$$
$$(T_1, T_2) \longmapsto [T_1, T_2]$$
satisfies:
1) It is bilinear;
2) (Antisymmetry) $[T_1, T_2] + [T_2, T_1] = 0$
3) (Jacobi) $[T_1, [T_2, T_3]] + [T_3, [T_1, T_2]] + [T_2, [T_3, T_1]] = 0.$

Remark 3.25
The Jacobi identity is a substitute of

associativity.

Associativity would amount to

$$[T_1,[T_2,T_3]] = [[T_1,T_2],T_3]$$
$$= -[T_3,[T_1,T_2]]$$

Hence $[T_1,[T_2,T_3]] + [T_3,[T_1,T_2]] = 0.$

$\boxed{\text{Definition } 3.26}$ [Lie algebra]

A Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ endowed with a map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, $(x,y)\mapsto[x,y]$ satisfying the properties 1), 2) and 3) above.

$\boxed{\text{Example } 3.27}$

1) If V is a $\mathbb{K}$-vector space then $\mathrm{End}(V)$ endowed with the bracket is a Lie algebra.

2). If M is a smooth manifold, then $\mathrm{Vect}(M)$ endowed with the bracket is a Lie algebra.

3) $\mathbb{R}^3$ with the cross product is a Lie algebra.

$\boxed{\text{Definition } 3.28}$ [Lie algebra homomorphism]

A $\mathbb{K}$-linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ of $\mathbb{K}$-Lie algebras is a Lie algebra homomorphism if $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ $\forall x,y \in \mathfrak{g}$.

Given a smooth map $\varphi : M \longrightarrow M'$ where M, M' are smooth manifolds in general there is no induced map $\mathrm{Vect}^\infty(M) \longrightarrow \mathrm{Vect}^\infty(M')$

However there is such induced map if we assume that φ is a diffeomorphism.

See pag. 26 below for the definition of Derivative !

More generally we can introduce the following.

Definition 3.29 [φ-related vector fields]

We say that $X \in \mathrm{Vect}^\infty(M)$ and $X' \in \mathrm{Vect}^\infty(M')$ are φ-related if

$$X'_{\varphi(m)} = D_m\varphi\,(X_m) \quad \forall m \in M.$$

There is a useful algebraic reformulation

Let $\varphi^*(f) := f \circ \varphi$, $f \in C^\infty(M')$

Then $\varphi^* : C^\infty(M') \longrightarrow C^\infty(M)$

is an algebra homomorphism.

Lemma 3.30

X and X' are φ-related iff the diagram

$$\begin{tikzcd} C^\infty(M') \arrow[r, "\varphi^*"] \arrow[d, "\alpha X'"'] & C^\infty(M) \arrow[d, "\alpha X"] \\ C^\infty(M') \arrow[r, "\varphi^*"] & C^\infty(M) \end{tikzcd}$$

commutes.

The proof is left as an Exercise.

Proposition 3.31

If X_i and X_i' are φ-related $i=1,2$, then $[X_1, X_2]$ and $[X_1', X_2']$ are φ-related.

Proof

By Lemma 3.30 above we have:

$$\varphi^*\alpha([X_1', X_2']) = \varphi^*(\alpha(X_1')\alpha(X_2') - \alpha(X_2')\alpha(X_1'))$$

↪ Def. 3.23

$$= \alpha(X_1)\varphi^*\alpha(X_2') - \alpha(X_2)\varphi^*\alpha(X_1')$$ Lemma 3.30

$$= \alpha(X_1)\alpha(X_2)\varphi^* - \alpha(X_2)\alpha(X_1)\varphi^*$$ Lemma 3.30

$$= (\alpha(X_1)\alpha(X_2) - \alpha(X_2)\alpha(X_1))\varphi^*$$

$$= \alpha([X_1, X_2])\varphi^*$$ Def 3.23

Hence $[X_1', X_2']$ and $[X_1, X_2]$ are φ-related by Lemma 3.30 again. (converse implication) $\square$

We note that if $\varphi : M \to M'$ is a diffeo then $\varphi^* : C^\infty(M') \to C^\infty(M)$ is an isomorphism of algebras.
Hence given $X \in \mathrm{Vect}^\infty(M)$ there is a unique $X' \in \mathrm{Vect}^\infty(M')$ which is φ-related to X, namely

$$\alpha X' = (\varphi^*)^{-1}\alpha X \varphi^* .$$

We will denote $X' := \varphi_* X$.

Corollary 3.32

If $\varphi : M \to M'$ is a diffeomorphism then

$$\mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M')$$
$$X \longmapsto \varphi_* X$$

is a <u>Lie algebra isomorphism</u>.

———— o ————

For the above definitions it is helpful to recall that: <u>the derivative, or tangent map</u>, at $p \in M$ of a smooth map $\varphi : M \to M'$ is defined in the following way.

Let $X_p : C^\infty(p) \to \mathbb{R}$ be a <u>tangent vector</u> and $f \in C^\infty(\varphi(p))$, with a slight abuse of notation let (U, f) be a representative with $U \ni p$ open.

check that it defines a tangent vector.

Then: $(D_p\varphi)(X_p)(f) := X_p(f \circ \varphi)$.

———— o ————

In the case when M is an <u>open subset</u> of a <u>finite dimensional vector space</u> over $\mathbb{R}$, V we will use some <u>conventions</u> and <u>identifications</u>.

Let $\Omega \subset V$ be open. We have the <u>identification</u> of the tangent space.

$$V \longrightarrow T_v\Omega \qquad v \in \Omega.$$
$$w \longmapsto w_v$$

where $w_v(f) = \frac{d}{dt}\Big|_{t=0} f(v+tw)$, $f \in C^\infty(\Omega)$

If then $L : V \longrightarrow U$ is any linear map.

$$(D_v L)(w_v) = \frac{d}{dt}\Big|_{t=0} L(v+tw) = L(w)$$

In particular, $\forall \lambda \in V^*$

$$D_v \lambda (w_v) = \lambda(w) .$$

Idea: • the tangent space of a vector space is the vector space itself.

• the tangent map of a linear map is the linear map itself.

See [Lee, Proposition 3.13] for more details.

$$(D_v L)(w_v)(f) = \frac{d}{dt}\Big|_{t=0} f(L(v+tw))$$

$$= \frac{d}{dt}\Big|_{t=0} f(Lv + tLw)$$

$$= Lw_{Lv}(f) .$$

3.3 The Lie algebra of a Lie group
Definition and examples

Let G be a Lie group and M a smooth manifold:

Definition 3.33 [Smooth action]
A left action of G on M is called smooth if the action map $G \times M \longrightarrow M$ is smooth.

If G acts smoothly on the left on M, then every $g \in G$ gives rise to a diffeomorphism

$$L_g : M \longrightarrow M.$$
$$x \longmapsto gx.$$

and hence, by Corollary 3.32 to a Lie algebra isomorphism

$$(L_g)_* : \mathrm{Vect}^\infty(M) \longrightarrow \mathrm{Vect}^\infty(M).$$

Definition 3.34 [G-invariant vector field]
A smooth vector field $X \in \mathrm{Vect}^\infty(M)$

is G-invariant if $\forall g \in G$ $(L_g)_* X = X$.

$\boxed{\text{Definition 3.35}}$ [Lie subalgebra]

Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra if $[X, Y] \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$.

By Corollary 3.32 the subvector space $\mathrm{Vect}^\infty(M)^G$ of G-invariant vector fields in $\mathrm{Vect}^\infty(M)$ is a Lie subalgebra of $\mathrm{Vect}^\infty(M)$.

Let now G act on the left on itself.

$$G \times G \longrightarrow G$$
$$(g, x) \longmapsto g \cdot x.$$

Then $\mathrm{Vect}_L^\infty(G)^G$ the space of left-invariant vector fields is a Lie algebra.

Moreover we have the following

$\boxed{\text{Lemma 3.36}}$

$$\begin{aligned} \mathrm{Vect}^\infty_L(G)^G &\longrightarrow T_e G \\ X &\longmapsto X_e. \end{aligned}$$

is a vector space isomorphism.

Proof

We define a map
$$\begin{aligned} T_e G &\longrightarrow \mathrm{Vect}^\infty_L(G) \\ v &\longmapsto v^L \end{aligned}$$

as follows

$$v^L_g := D_e L_g(v).$$

The fact that $v^L \in \mathrm{Vect}^\infty_L(G)^G$ follows from the chain rule.

Note also that $v^L_e = v$ since $L_e = \mathrm{id}_G$.

On the other hand, if $X \in \mathrm{Vect}^\infty_L(G)^G$ then in particular

$$X_g = (D_e L_g).(X_e)$$

and hence $X = (X_e)^L$ $\square$

We are ready to introduce the definition of Lie algebra of a Lie group :

Definition 3.37 [Lie algebra of a Lie group]
The Lie algebra $\mathfrak{g}$ of a Lie group G is the vector space $\mathfrak{g} = T_e G$ endowed with the bracket $[v,w] = [v^L, w^L]_e$. $\forall\, v,w \in T_e G$

We would like to identify explicitly the Lie algebra of $GL(n,\mathbb{R})$. Recall that since $GL(n,\mathbb{R}) \subset M_{n,n}(\mathbb{R})$ is open we have the identification

$$\begin{aligned} M_{n,n}(\mathbb{R}) &\longrightarrow T_I\, GL(n,\mathbb{R}) \\ A &\longmapsto A_I \end{aligned}$$

Let us denote $\mathfrak{gl}(n,\mathbb{R})$ the Lie algebra of $GL(n,\mathbb{R})$ and for convenience

$$\tilde{A} = (A_I)^L$$

the left-invariant vector field corresponding to A_I

Then we have:

Proposition 3.38

The map

$$M_{n,n}(\mathbb{R}) \longrightarrow \mathfrak{gl}(n,\mathbb{R})$$
$$A \longmapsto \tilde{A}$$

induces an isomorphism between the Lie algebra $M_{n,n}(\mathbb{R})$ with matrix bracket and the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$. Equivalently

$$[\tilde{A},\tilde{B}] = \widetilde{[A,B]} \quad \forall A,B \in M_{n,n}(\mathbb{R})$$

Proof

Since both $[\tilde{A},\tilde{B}]$ and $\widetilde{[A,B]}$ are left-invariant vector fields it suffices to check that:

$$[\tilde{A},\tilde{B}]_I = \widetilde{[A,B]}_I .$$

By the identifications that we discussed a few pages ago two tangent vectors coincide iff their evaluation on all $\lambda \in M_{n,n}(\mathbb{R})^*$ do.

Therefore it is sufficient to show

$$[\tilde{A},\tilde{B}]_I(\lambda) = \widetilde{[A,B]}_I(\lambda)$$

$\forall \lambda \in M_{n \times n}(\mathbb{R})^*$

Note that $\widetilde{[A,B]}_I = [A,B]$

Hence recalling also that linear functions can be identified with their derivatives we need to show that

$$\lambda([A,B]) = [\tilde{A},\tilde{B}]_I(\lambda)$$

However $\lambda([A,B]) = \lambda(AB) - \lambda(BA)$ since $[A,B]$ is the bracket in $M_{n \times n}(\mathbb{R})$.

On the other hand:

$$[\tilde{A},\tilde{B}]_I(\lambda) = (\tilde{A}\tilde{B} - \tilde{B}\tilde{A})(\lambda)(I)$$

We proceed to show that:

$$\tilde{A}\tilde{B}(\lambda)(I) = \lambda(AB).$$

This will be enough to complete the proof.

$$\tilde{A}\tilde{B}(\lambda)(I) = A_I(\tilde{B}(\lambda)).$$

$$= A_I(g \longmapsto \tilde{B}_g(\lambda))$$

$$= A_I(g \longmapsto D_I L_g(B_I).(\lambda)).$$

But $D_I L_g(B_I)(\lambda) = B_I(h \longmapsto \lambda(gh))$

Furthermore. $h \longmapsto \lambda(gh)$. is the restriction of a linear form in $M_{n,n}(\mathbb{R})$ to $GL(n,\mathbb{R})$

Hence. $B_I(\lambda \longmapsto \lambda(gh)) = \lambda(gB)$.

Hence $\tilde{A}\tilde{B}(\lambda)(I) = A_I(g \longmapsto \lambda(gB))$ and for the same reasons as above.

$A_I(g \longmapsto \lambda(gB)) = \lambda(AB)$ as claimed

$\square$

Our next goal will be to understand whether a smooth homomorphism of Lie.

groups induces a Lie algebra homomorphism.

We have the following

Proposition 3.39

Let $\phi : G \longrightarrow H$ be a smooth homomorphism of Lie groups and $\mathfrak{g} = T_e G$ and $\mathfrak{h} = T_e H$ be their Lie algebras. Then $D_e\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism.

Proof

Let $v \in T_e G$, $v^L \in \mathrm{Vect}^\infty(G)^G$ the corresponding left invariant vector field. $w := D_e\phi(v) \in T_e H$ and $w^L \in \mathrm{Vect}^\infty(H)^H$

Claim: v^L and w^L are ϕ-related.

To prove the claim we note that

$$w^L_{\phi(g)} \overset{\text{def of } w^L}{=} D_e L_{\phi(g)}(w) \overset{\text{def of } w}{=} D_e L_{\phi(g)} D_e\phi(v)$$

$$= D_e\left(L_{\varphi(g)} \circ \varphi\right)(v).$$

Chain rule

Since φ is a homomorphism

$$L_{\varphi(g)} \circ \varphi = \varphi \circ L_g$$

Hence.

Chain rule again.

$$= D_e\left(\varphi \circ L_g\right)(v) = D_g\varphi\left(D_e L_g(v)\right)$$

$$= D_g\varphi\left(v_g^L\right)$$

Def v^L

Thus if $v_1, v_2 \in T_eG$ and $w_i := D_e\varphi(v_i)$ then since v_i^L and w_i^L are φ-related, it follows from Proposition 3.31 that $[v_1^L, v_2^L]$ and $[w_1^L, w_2^L]$ are φ-related.

Hence

$$D_e\varphi([v_1, v_2]) = D_e\varphi\left([v_1^L, v_2^L]_e\right)$$

$$= [w_1^L, w_2^L]_e$$

$$= [w_1, w_2]$$

$$= [D_e\phi(v_1), D_e\phi(v_2)]$$

$\square$

Corollary 3.40

Let G be a Lie group and $H < G$ be a subgroup which is also a regular submanifold. Then the inclusion $H \longrightarrow G$ realizes $\mathfrak{h} = T_e H$ as a Lie subalgebra of $\mathfrak{g} = T_e G$

Example 3.41

1) The Lie algebra of $SL(n,\mathbb{R})$ is:

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in M_{n,n}(\mathbb{R}) : \operatorname{tr} X = 0\}$$

(cf with Example 3.14 1)).

Indeed we saw that $SL(n,\mathbb{R}) = \det^{-1}(1)$ and $\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$ has constant rank with $(D_I \det)(X) = \operatorname{tr} X$

Therefore if $\gamma : (-\varepsilon, \varepsilon) \longrightarrow SL(n,\mathbb{R})$ is a smooth curve $\frac{d}{dt}\det(\gamma(t)) = 0$

If we choose γ such that $\gamma(0) = 1$ and so.

$\gamma'(0) \in T_I\, SL(n,\mathbb{R}) = sl(n,\mathbb{R})$ then

$$0 = \frac{d}{dt}\Big|_{t=0} \det(\gamma(t)) = D_I \det(\gamma'(0))$$

$$= \operatorname{tr}(\gamma'(0))$$

Hence $sl(n,\mathbb{R}) \subseteq \{A \in gl(n,\mathbb{R}) : \operatorname{tr} A = 0\}$

Since $\dim sl(n,\mathbb{R}) = \dim \{A \in gl(n,\mathbb{R}) : \operatorname{tr} A = 0\}$

equality in the above inclusion holds.

2) The Lie algebra of $O(n,\mathbb{R})$ is
$o(n,\mathbb{R}) = \{X \in M_{n,n}(\mathbb{R}) : X + {}^tX = 0\}$
(cf. with Example 3.14 2)).

For checking the above it is helpful to keep in mind the following:

If $A, B : (-\varepsilon, \varepsilon) \longrightarrow M_{n,n}(\mathbb{R})$ are smooth curves and we set
$p(s) := A(s) \cdot B(s) \in M_{n,n}(\mathbb{R})$

then p is a smooth curve and

$$p'(s) = A'(s)\,B(s) + A(s)\,B'(s).$$

3) Note that $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ is a subgroup and a regular submanifold of $GL(n,\mathbb{R})$. Its Lie algebra is

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}.$$

Analogously for $A = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}_{>0} \right\}$.

we have

$$\mathfrak{a} = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} : x_i \in \mathbb{R} \right\}$$

Note that $[\,,\,]$ vanishes on $\mathfrak{a}$.

$\boxed{\text{Exercise 3.42.}}$

1) Compute the Lie algebra of $O(p,q)$ and $SO(p,q)$ for $p+q=n$.

2) Realise. $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $U(n)$ as Lie groups and compute their Lie algebras.

Example 3.43

Let G, H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with bracket

$$[(x_1,y_1),(x_2,y_2)] = ([x_1,x_2]_{\mathfrak{g}}, [y_1,y_2]_{\mathfrak{h}}).$$

(Exercise).

In abstract terms Proposition 3.39 says that we have constructed a <u>functor</u>

$$\text{Lie} : \text{Lie groups} \longrightarrow \text{Lie algebras}.$$

(a "morphism" (= Lie group homomorphism) between Lie groups naturally induces a morphism (= Lie algebra homomorphism) between the respective Lie algebras).

The fundamental question is how much information we loose by going from Lie groups to Lie algebras.

Some informal remarks follow. We will clarify some of them over the next few lectures.

1) Every Lie algebra is the Lie algebra of a Lie group. More generally we shall discuss the "Lie group - Lie algebra correspondence".
2) [Faithfulness] Note that if G is a Lie group and F is any finite group with the discrete topology then G and $G \times F$ have the same Lie algebra.

It might seem that this is related to disconnectedness. However, even if G is connected, it is not uniquely determined by its Lie algebra.

For instance, we note that

$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ is a covering map and it is easy to see that it induces an isomorphism between the invariant vector fields and hence, between the Lie algebras.

In fact if G_1 and G_2 are connected Lie groups then any isomorphism,

$$\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$$

comes from an isomorphism $\tilde{G}_1 \to \tilde{G}_2$ (we will prove this later).

3). It would be nice if the category of Lie groups is closed under certain natural operations like taking the center $Z(G)$ of a Lie group G, or G° the connected component of the identity,

In this direction we will see a very important theorem due to Cartan saying that if $H < G$ is a closed

subgroup then it is a regular
submanifold and hence a Lie group.