Structure theory of spaces with lower Ricci bounds towards codimension one

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Introduction

RCD(K,N) spaces are a non smooth spaces with:

- Ricci curvature bounded from below by K;
- dimension bounded from above by N,

and looking infinitesimally like Riemannian manifolds (rather than Finsler ones).

Developing a theory of codimension one hypersurfaces in this non smooth framework.

Relying on the theory of sets of finite perimeter, pioneered by Caccioppoli and De Giorgi.

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Outline

References

- E. Brué, E. Pasqualetto, D. Semola; Rectifiability of $\mathrm{RCD}(K, N)$ spaces via δ splitting maps, accepted by Ann. Acad. Sci. Fenn. Math., (2020),
- L. Ambrosio, E. Brué, D. Semola; Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces. Geom. Funct. Anal. (2019),
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Ricci curvature

Let us consider a smooth Riemannian manifold (M,g), $p\in M$, $u,v\in T_pM$ orthonormal vectors.
The sectional curvature $K(u,v)$ governs distortions of distances

$$\mathrm{d}(\exp_p(tu),\exp_p(tv)) = \sqrt{2}t\left(1-\frac{K(u,v)}{12}t^2+O(t^3)\right) \quad \text{as } t\to 0.$$

Ricci curvature is obtained averaging the sectional curvatures:

$$\mathrm{Ric}(u,u) := \sum_{i=2}^{n} K(u,e_i).$$

In harmonic coordinates (x^i):

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms}.$$

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Ricci curvature: Lagrangian vs Eulerian

Consider a smooth map $\psi : M \to \mathbb{R}$ and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t} T_t(x)$ and $\mathcal{J}(t) = \det DT_t(x)$ is the volume element,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma, \dot\gamma)}{n}\mathcal{J}^{1/n} \le 0 \tag{1}$$

and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \ge \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi, \nabla\psi). \tag{2}$$

(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

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Ricci curvature bounds

Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;
- Cheeger-Gromoll splitting theorem;
- Li-Yau heat kernel bounds;
- Lévy-Gromov isoperimetric inequality.

Theorem

The class $\mathcal{M}_{N,D,K}$ of smooth Riemannian manifolds with dimension N, diameter bounded above by D and Ricci curvature bounded below by K is precompact w.r.t. the Gromov-Hausdorff topology.

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How do Riemannian manifolds in $\mathcal{M}_{N,D,K}$ look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in $\mathcal{M}_{N,D,K}$, initiated by Cheeger-Colding in the Nineties.

The study of Ricci limits improves our knowledge of $\mathcal{M}_{N,D,K}$:

- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform L^2 bounds for the Riemann curvature tensor ([Jiang-Naber '16]);
- gradients of harmonic functions do not verify uniform continuity estimates on Riemannian manifolds with lower Ricci curvature bounds ([Cheeger-Naber '15], [De Philippis-Zimbron '19]).

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The synthetic treatment of lower Ricci curvature bounds stems from the following

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Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Analogy with the theory of Alexandrov spaces, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with metric measure spaces. The heat flow could play a role.
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$\mathrm{RCD}(K, N)$ spaces

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- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;
- [Sturm '06] and [Lott-Villani '07], for the proposal of the $\mathrm{CD}(K, N)$ Curvature-Dimension condition on metric measure spaces;
- [Ohta-Sturm '09], [Ambrosio-Gigli-Savaré '12], [Gigli '13] for the additional Riemannian assumption.

Definition ($\mathrm{RCD}(K, N)$ m.m.s.)

For any $K \in \mathbb{R}$ and $1 \le N < \infty$ we say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, N)$ if it is an infinitesimally Hilbertian $\mathrm{CD}(K, N)$ m.m.s.

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- The splitting theorem ([Cheeger-Gromoll], [Cheeger-Colding]) holds for $\mathrm{RCD}(0, N)$ spaces [Gigli '13]. The split factor is $\mathrm{RCD}(0, N-1)$;
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Tangent spaces

Question

How regular is an $\mathrm{RCD}(K,N)$ space?

Definition (Tangent cone)

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ and $x\in X$ we let $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ be the set of all pmGH limits

$$(Y,\mathsf{d}_Y,\mathfrak{m}_Y,y)=\lim_{i\to\infty}(X,r_i^{-1}\mathsf{d},\mathfrak{m}^x_{r_i},x),$$

where $r_i\downarrow 0$ and $\mathfrak{m}^x_{r_i}=c^x_{r_i}\mathfrak{m}$ for some normalizing constant $c^x_{r_i}>0$.

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Structure theory up to measure zero

Definition (k-regular set)

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s.. For any $1 \le k \le N$ let

$$\mathcal{R}_k := \left\{ x \in X : \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \left\{ \left(\mathbb{R}^k, \mathsf{d}_{eucl}, c_k \mathscr{L}^k, 0^k \right) \right\} \right\} .$$

After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

Theorem

It holds

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k \right) = 0 .$$

Furthermore, for any $1 \le k \le N$ the k-regular set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$, for some $\theta \in L^1_{loc}(\mathcal{H}^k \llcorner \mathcal{R}_k)$.

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Constancy of the dimension

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Let $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s.. Then there exists* $1 \le n \le N$ *such that*

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The same statement was proved in for Ricci limit spaces, via a different strategy.
Their strategy has been recently generalized to the RCD framework in [Deng '20].

In [Brué-S. '18] we obtain new regularity estimates for flows of Sobolev vector fields, valid beyond the case needed for proving constancy of the dimension.

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Definition (Harmonic δ-splitting map)

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So far we have a good understanding of the structure of $\mathrm{RCD}(K, N)$ spaces up to $\mathfrak{m}$-negligible sets.

Aim

Pushing the study of $\mathrm{RCD}(K, N)$ spaces up to codimension one subsets.

In the last couple of years:

- rectifiability of singular sets on non collapsed Ricci limits [Cheeger-Jiang-Naber '18];
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Sets of finite perimeter

Theory pioneered by Caccioppoli and De Giorgi in the 1950s.

Remark

On $\mathbb{R}^n$, sets of finite perimeter are Borel sets E such that the distributional derivative $D\chi_E$ is a finite (vector valued) measure.

Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and $f \in L^1_{loc}(X, \mathfrak{m})$. Given an open set $A \subset X$ we let

$$|Df|(A) := \inf \left\{ \liminf_{n\to\infty} \int_A \mathrm{lip} f_n \, \mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_{loc}(A), f_n \to f \text{ in } L^1_{loc}(A, \mathfrak{m}) \right\}.$$

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On $\mathbb{R}^n$, sets of finite perimeter are Borel sets E such that the distributional derivative $D\chi_E$ is a finite (vector valued) measure.

Let $(X, \mathrm{d}, \mathfrak{m})$ be a m.m.s. and $f \in L^1_{loc}(X, \mathfrak{m})$. Given an open set $A \subset X$ we let

$$|Df|(A) := \inf \left\{ \liminf_{n\to\infty} \int_A \mathrm{lip} f_n \, \mathrm{d}\mathfrak{m} : f_n \in \mathrm{Lip}_{loc}(A), f_n \to f \text{ in } L^1_{loc}(A, \mathfrak{m}) \right\} .$$

Definition ([Miranda jr. '03])

We say that $f \in L^1(X, \mathfrak{m})$ has bounded variation if $|Df|(X) < \infty$. In that case $|Df|$ is the restriction to open sets of a finite Borel measure.

$E \subset X$ has finite perimeter if χ_E has bounded variation.

The Euclidean theory

Definition (Reduced boundary)

Given a set of finite perimeter $E \subset \mathbb{R}^n$, the *reduced boundary* $\mathcal{F}E$ is the set of those $x \in \mathbb{R}^n$ such that there exists

$$\nu_E(x) := \lim_{r \downarrow 0} \frac{D\chi_E\,(B_r(x))}{|D\chi_E|\,(B_r(x))} \quad \text{and } |\nu_E(x)| = 1.$$

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By Besicovitch's theorem, $|D\chi_E|$ is concentrated on $\mathcal{F}E$ and $D\chi_E = \nu_E |D\chi_E|$.

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De Giorgi's theorem

Theorem ( )

Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then:

Question

Is it possible to formulate and prove an analogue of De Giorgi's theorem in the RCD context?

De Giorgi's theorem

Theorem ([De Giorgi '55])

Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then:

- *for any $x \in \mathcal{F}E$, the rescaled sets $E_{x,r} = \frac{E-x}{r}$ converge as $r \downarrow 0$ in $L^1_{loc}(\mathbb{R}^n)$ to the half-space*

$$H = \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\};$$

- *for any $x \in \mathcal{F}E$ it holds*

$$\lim_{r \to 0} \frac{|D\chi_E|(B_r(x))}{\omega_n r^{n-1}} = 1;$$

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In [Ambrosio-Brué-S. '18] and [Brué-Pasqualetto-S. '19] we generalized De Giorgi's theorem to $\mathrm{RCD}(K,N)$ spaces.

We obtain:

- existence of regular blow-ups almost everywhere w.r.t. the perimeter;
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Blow-ups of sets of finite perimeter

Remark

The analysis of blow-ups is the starting point in the Euclidean case.

Definition (Tangent to a set of finite perimeter)

Let (X, d, m) be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ be a set of locally finite perimeter. We let $\mathrm{Tan}_x(X, d, m, E)$ be the collection of all quintuples (Y, ϱ, μ, y, F) such that

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Motivations & remarks

- There are lots of sets of finite perimeter, thanks to the coarea formula.
- The strategy via Besicovitch's theorem cannot be easily adapted.

- In this way results about sets of finite perimeter turn into results about $\mathrm{RCD}(K, N)$ spaces.
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Let (X, d, m) be an RCD$(0, N)$ m.m.s. of finite measure, $\mathbb{R} = (\mathbb{R}, d_{eucl}, \mathscr{L}^1)$ and let $Y = X \times \mathbb{R}$ be the product space, with product metric and product measure.

If we let $E := \{t \geq 0\}$, then E is of finite perimeter and $|D\chi_E|$ can be identified with m, up to identification of X with $X \times \{0\}$.

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Regularity of the reduced boundary, I

Theorem ( )

Let $E \subset X$ be of finite perimeter. Then for $|D\chi_E|$-a.e. $x \in X$ there exists a unique $1 \le k \le n$ such that

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Let now $\mathcal{F}_k E$ be the set of points with k-dimensional Euclidean blow-up.

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Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ a set of finite perimeter. Then, for any $1 \le k \le n$, it holds that $\mathcal{F}_k E$ is $(|D\chi_E|, k-1)$-rectifiable.

Boundaries of sets of finite perimeter have the same rectifiable regularity of the ambient spaces.

The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

A key step is the construction of a suitable unit normal vector field.

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The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

A key step is the costruction of a suitable *unit normal vector field*.

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Theorem ([Brué-Pasqualetto-S. '19])

Let $(X, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ a set of finite perimeter. Then, for any $1 \le k \le n$, it holds that $\mathcal{F}_k E$ is $(|D\chi_E|, k-1)$-rectifiable.

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Open problems, I

Definition

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ we let $\dim X$ be the unique natural $1 \le n \le N$ such that $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$.

Open problem

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ does it hold $\dim X = \dim_H X$?

Open problem

Given an $\mathrm{RCD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ with $\dim X = n$, is there an open set $U \subset X$ homeomorphic to a smooth n-dimensional manifold and with $\mathfrak{m}(X \setminus U) = 0$?

These problems are open even for collapsed Ricci limits, [Colding-Naber '12], [Naber '14, '20].

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Studying sets of finite perimeter:

Conjecture

Given an $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ with $\dim X = n$ and a set of finite perimeter $E \subset X$, the tangent space to X has constant dimension (equal to n) $|D\chi_E|$-almost everywhere.

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Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space with $\dim X = n$. Let $E \subset X$ be a set of finite perimeter. Then $\mathcal{F}_n E$ has σ-finite $\mathcal{H}^{n-1}$ measure and $|D\chi_E| \ll \mathcal{H}^{n-1} \llcorner \mathcal{F}_n E$.

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Thank you for your attention!