

Structure theory  
of  $\text{RCD}(K, N)$   
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Ricci curvature

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spaces

Structure theory  
up to negligible  
sets

De Giorgi's  
theorem on  
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Open problems

# Structure theory of spaces with lower Ricci bounds towards codimension one

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# Introduction

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$\text{RCD}(K, N)$  spaces are a non smooth spaces with:

- Ricci curvature bounded from below by  $K$ ;
- dimension bounded from above by  $N$ ,

and looking infinitesimally like Riemannian manifolds (rather than Finsler ones).

Developing a theory of codimension one hypersurfaces in this non smooth framework.

Relying on the theory of sets of finite perimeter, pioneered by Caccioppoli and De Giorgi.

• A theory independent of specific analytical assumptions?

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- **dimension** bounded from **above** by  $N$ ,

and looking infinitesimally like **Riemannian** manifolds (rather than Finsler ones).

Developing a theory of **sets of finite perimeter** by generalizing the notion of smooth manifolds.

Relying on the theory of **sets of finite perimeter**, pioneered by **Caccioppoli** and **De Giorgi**.

Structure theory of  $\text{RCD}(K, N)$  spaces by Semola and Soligo

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Developed by Semola and Tamura, based on the theory of sets of finite perimeter.

Relying on the theory of **sets of finite perimeter**, pioneered by **Caccioppoli** and **De Giorgi**.

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*... a theory independent of special topological investigations!*

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Relying on the theory of **sets of finite perimeter**, pioneered by **Caccioppoli** and **De Giorgi**.

*...a theory independent of special topological investigations<sup>1</sup>*

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<sup>1</sup>from the review of L. C. Young to **[Caccioppoli '51]**.

# Outline

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- **E. Brué, E. Pasqualetto**, D. Semola; Rectifiability of  $\text{RCD}(K, N)$  spaces via  $\delta$  splitting maps, accepted by Ann. Acad. Sci. Fenn. Math., (2020),
- L. Ambrosio, E. Brué, D. Semola; Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces. Geom. Funct. Anal. (2019),
- E. Brué, E. Pasqualetto, D. Semola; Rectifiability of the reduced boundary for sets of finite perimeter over  $\text{RCD}(K, N)$  spaces, preprint (2019).

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Let us consider a smooth Riemannian manifold  $(M, g)$ ,  $p \in M$ ,  $u, v \in T_p M$  orthonormal vectors.

The sectional curvature  $K(u, v)$  governs distortions of distances

$$d(\exp_p(tu), \exp_p(tv)) = \sqrt{2}t \left( 1 - \frac{K(u, v)}{12}t^2 + O(t^3) \right) \quad \text{as } t \rightarrow 0.$$

Ricci curvature is obtained averaging the sectional curvatures:

$$\text{Ric}(u, u) := \sum_{i=2}^n K(u, e_i).$$

What is the Ricci curvature of  $\mathbb{Q}^n$ ?

What is the Ricci curvature of  $\mathbb{R}^n$  with the standard metric?

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In harmonic coordinates  $(x^i)$ :

$$\text{Ric}_i = -\frac{1}{2} \Delta g_i + \text{lower order terms.}$$

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## Remark

In harmonic coordinates  $(x^i)$ :

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# Ricci curvature: Lagrangian vs Eulerian

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Consider a smooth map  $\psi : M \rightarrow \mathbb{R}$  and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if  $\dot{\gamma} := \frac{d}{dt}T_t(x)$  and  $\mathcal{J}(t) = \det DT_t(x)$  is the volume element,

$$\frac{d^2}{dt^2}(\mathcal{J}(t))^{1/n} + \frac{\text{Ric}(\dot{\gamma}, \dot{\gamma})}{n} \mathcal{J}^{1/n} \leq 0 \quad (1)$$

and

$$\Delta \frac{|\nabla\psi|^2}{2} - \nabla\psi \cdot \nabla\Delta\psi \geq \frac{(\Delta\psi)^2}{n} + \text{Ric}(\nabla\psi, \nabla\psi). \quad (2)$$

(1) is a Lagrangian formulation of (2) (the Eulerian one) and is more useful in the study of the evolution of volume and perimeter.

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(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

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Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov volume inequality on monotonicity of volume ratios;
- Cheeger-Colding collapsing theorem;
- Li-Yau heat kernel bounds;
- Li-Yang-Yang isoperimetric inequality;

• **Open problems:** rigidity of Bishop-Gromov inequality, rigidity of Cheeger-Colding theorem, rigidity of Li-Yau heat kernel bounds, rigidity of Li-Yang-Yang isoperimetric inequality, rigidity of De Giorgi's theorem on  $\text{RCD}(K, N)$  spaces, rigidity of the structure theory up to negligible sets, rigidity of the structure theory of  $\text{RCD}(K, N)$  spaces.

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Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;

• Cheeger-Croke inequality on volume and diameter;

• Bonnet-Myer inequality on volume and Ricci curvature;

• Bishop-Miranda inequality on volume and Ricci curvature;

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- **Bishop-Gromov** inequality on monotonicity of volume ratios;
- **Cheeger-Gromoll** splitting theorem;
- **Li-Yau** heat kernel bounds;
- **Lévy-Gromov** isoperimetric inequality.

## Theorem ([Gromov '82])

*The class  $\mathcal{M}_{N,D,K}$  of smooth Riemannian manifolds with dimension  $N$ , diameter bounded above by  $D$  and Ricci curvature bounded below by  $K$  is precompact w.r.t. the Gromov-Hausdorff topology.*

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## Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- **Bishop-Gromov** inequality on monotonicity of volume ratios;
- **Cheeger-Gromoll** splitting theorem;
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- **Lévy-Gromov** isoperimetric inequality.

Theorem ([Gromov '82])

*The class  $\mathcal{M}_{N,D,K}$  of smooth Riemannian manifolds with dimension  $N$ , diameter bounded above by  $D$  and Ricci curvature bounded below by  $K$  is precompact w.r.t. the Gromov-Hausdorff topology.*

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How do Riemannian manifolds in  $\mathcal{M}_{N,D,K}$  look like?

The question motivated the theory of **Ricci limits**, limits in the (pm)GH topology of manifolds in  $\mathcal{M}_{N,D,K}$ , initiated by **Cheeger-Colding** in the Nineties.

The study of Ricci limits improves our knowledge of  $\mathcal{M}_{N,D,K}$ :

- Cheeger-Colding manifolds with bounded Ricci curvature and volume lower bound have very uniform  $L^2$ -bounds for the eigenfunctions of the Laplacian
- the gradients of harmonic functions do not vary uniformly
- the eigenvalues of the Laplacian are asymptotically distributed with respect to the Weyl law

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- Riemannian manifolds with bounded Ricci curvature and volume bounded from below verify uniform  $L^2$ -bounds for the Riemann curvature tensor (Jiang-Naber'16);

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The **synthetic** treatment of lower Ricci curvature bounds stems from the following

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

It coincides with the theory of **lower Ricci curvature bounds** in the synthetic setting.

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(Bridson '07): such a theory should deal with metric measure spaces. The heat flow could play a role.

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## Remark

Analogy with the theory of **Alexandrov spaces**, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with **metric measure spaces**. The **heat flow** could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

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After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;

• [Cordero-Erausquin-McCann-Schmuckenschläger '01], [McCann '97], [Otto-Villani '00], for the proposal of the RCD( $K, N$ ) synthetic curvature condition on metric measure spaces;

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For any  $K \in \mathbb{R}$  and  $1 \leq N < \infty$  we say that  $(X, d, m)$  is RCD( $K, N$ ) if it is an infinitesimally Hilbertian CD( $K, N$ ) m.m.s.

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After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

A space  $(X, d, \mu)$  is  $\text{RCD}(K, N)$  if and only if

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## Theorem ( $\text{RCD}^*(K, N)$ spaces)

*A m.m.s.  $(X, d, m)$  is  $\text{RCD}^*(K, N)$  if:*

*$m(B_r(x)) \leq c_1 \exp(c_2 r^2)$  for some  $x \in X$  and constants  $c_1, c_2 > 0$ ;*

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- it is infinitesimally Hilbertian ( $W^{1,2}(X, d, m)$  is Hilbert);
- it satisfies the Sobolev to Lipschitz property;
- $\Delta \frac{1}{2} |\nabla f|^2 - \nabla f \cdot \nabla \Delta f \geq \frac{(\Delta f)^2}{N} + K |\nabla f|^2$ , for any  $f$  in a class of test functions.

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# The Eulerian approach

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Ricci curvature

$\text{RCD}(K, N)$   
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De Giorgi's  
theorem on  
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Open problems

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

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$\text{RCD}^*(K, N)$  is equivalent to  $\text{RCD}(K, N)$  if  $m(X) < \infty$ , thanks to Cavalletti-Moran '16.

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- cones over  $\text{RCD}(N - 2, N - 1)$  spaces are  $\text{RCD}(0, N)$  [Ketterer '13], quotients of  $\text{RCD}^*(K, N)$  spaces under suitable group actions are  $\text{RCD}^*(K, N)$  spaces, [Galaz-Garcia-Kell-Mondino-Sosa '17];
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How regular is an  $\text{RCD}(K, N)$  space?

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  and  $x \in X$  we let  
 $\text{Tan}_x(X, d, m)$  be the set of all pmGH limits

$$(Y, d_Y, m_Y, \gamma) = \lim_{r \rightarrow \infty} (X, r^{-1}d, m_r^x, x),$$

where  $r \downarrow 0$  and  $m_r^x = c_r^x m$  for some normalizing constant  $c_r^x > 0$ .

If  $(X, d, m) = (M^n, g, m_g)$ , then

$$\text{Tan}_x(X, d, m) = \{(M^n, g_{\text{eucl}}, c_r^x m_g)\} \text{ for any } x \in M^n.$$

What happens if  $(X, d, m)$  is not a metric space or if  $X$  is not the same limit?

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Open problems

## Question

How regular is an  $\text{RCD}(K, N)$  space?

## Definition (Tangent cone)

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  and  $x \in X$  we let  $\text{Tan}_x(X, d, m)$  be the set of all pmGH limits

$$(Y, d_Y, m_Y, y) = \lim_{i \rightarrow \infty} (X, r_i^{-1}d, m_{r_i}^x, x),$$

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- If  $(X, d, m) = (M^n, d_g, \text{vol}_g)$ , then  $\text{Tan}_x(X, d, m) = \{(\mathbb{R}^n, d_{\text{eucl}}, c_n \mathcal{L}^n, 0^n)\}$  for any  $x \in X$ ;
- the tangent cone to a metric cone at its tip is the cone itself.

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Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. For any  $1 \leq k \leq N$  let

$$R_k := \{x \in X : \text{Tan}_x(X, d, m) = \{(\mathbb{R}^k, d_{\text{euc}}, c_x \llcorner \mathcal{L}^k, 0^k)\}\}.$$

After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

$\mathcal{H}^k \llcorner R_k = 0$  for any  $1 \leq k \leq N$  and  $\mathcal{H}^k \llcorner R_k = 0$  for any  $1 \leq k \leq N$ .

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## Definition ( $k$ -regular set)

Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  m.m.s.. For any  $1 \leq k \leq N$  let

$$\mathcal{R}_k := \left\{ x \in X : \text{Tan}_x(X, d, \mathfrak{m}) = \left\{ (\mathbb{R}^k, d_{\text{eucl}}, c_k \mathcal{L}^k, 0^k) \right\} \right\}.$$

After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

*It holds*

$$\mathfrak{m} \left( X \setminus \bigcup_{k=1}^N \mathcal{R}_k \right) = 0.$$

*Furthermore, for any  $1 \leq k \leq N$  the  $k$ -regular set  $\mathcal{R}_k$  is  $(\mathfrak{m}, k)$ -rectifiable and  $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$ , for some  $\theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$ .*

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## Theorem

*It holds*

$$m \left( X \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k \right) = 0.$$

*Furthermore, for any  $1 \leq k \leq N$  the  $k$ -regular set  $\mathcal{R}_k$  is  $(m, k)$ -rectifiable and  $m \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$ , for some  $\theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$ .*

# Constancy of the dimension

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*Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. Then there exists  $1 \leq n \leq N$  such that*

$$m(X \setminus \mathcal{R}_n) = 0.$$

*This strategy has been recently generalized to the RCD framework in [Brué-S. '18].*

In [Brué-S. '18] we obtain new regularity estimates for flows of Sobolev vector fields, valid beyond the case needed for proving constancy of the dimension.

# Constancy of the dimension

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## Theorem ([Brué-S. '18])

*Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. Then there exists  $1 \leq n \leq N$  such that*

$$m(X \setminus \mathcal{R}_n) = 0.$$

## Remark

The same statement was proved in [Colding-Naber '12] for Ricci limit spaces, via a different strategy.

In [Brué-S. '18] we obtain new regularity estimates for flows of Sobolev vector fields, valid beyond the case needed for proving constancy of the dimension.

# Constancy of the dimension

Structure theory  
of  $\text{RCD}(K, N)$   
spaces

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Ricci curvature

$\text{RCD}(K, N)$   
spaces

Structure theory  
up to negligible  
sets

De Giorgi's  
theorem on  
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Open problems

## Theorem ([Brué-S. '18])

*Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. Then there exists  $1 \leq n \leq N$  such that*

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## Definition (Harmonic $\delta$ -splitting map)

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s. and  $B_r(x) \subset X$ . We say that  $u : B_r(x) \rightarrow \mathbb{R}^k$  is a  $\delta$ -splitting map provided:

1.  $u$  has harmonic and Lipschitz components;

2.  $u$  is  $\delta$ -harmonic on  $B_{r/2}(x)$ .

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- $u$  has harmonic and Lipschitz components;

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In [Brué-Pasqualetto-S. 20] we gave simplified proofs of the structure theorems for  $\text{RCD}(K, N)$  spaces using harmonic  $\delta$ -splitting maps to rectify.

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## Harmonic $\delta$ -splitting maps and structure theory

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Open problems

So far we have a good understanding of the structure of  $RCD(K, N)$  spaces up to  $m$ -negligible sets.

Pushing the study of  $RCD(K, N)$  spaces up to codimension one subsets.

In the last couple of years:

• the rectifiability of  $RCD(K, N)$  spaces up to  $m$ -negligible sets

• the structure of boundary of non collapsed  $RCD$  spaces

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# Sets of finite perimeter

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Open problems

Theory pioneered by **Caccioppoli** and **De Giorgi** in the 1950s.

On  $\mathbb{R}^n$ , sets of finite perimeter are Borel sets  $E$  such that the distributional derivative  $D\chi_E$  is a finite (vector valued) measure.

Let  $(X, d, m)$  be a m.m.s. and  $f \in L^1_{loc}(X, m)$ . Given an open set  $A \subset X$  we let

$$|Df|(A) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_A \text{lip} f_n \, dm : f_n \in \text{Lip}_{loc}(A), f_n \rightarrow f \text{ in } L^1_{loc}(A, m) \right\}.$$

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On  $\mathbb{R}^n$ , sets of finite perimeter are Borel sets  $E$  such that the distributional derivative  $D\chi_E$  is a finite (vector valued) measure.

Let  $(X, d, m)$  be a m.m.s. and  $f \in L^1_{loc}(X, m)$ . Given an open set  $A \subset X$  we let

$$|Df|(A) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_A \text{lip} f_n \, dm : f_n \in \text{Lip}_{loc}(A), f_n \rightarrow f \text{ in } L^1_{loc}(A, m) \right\}.$$

# Sets of finite perimeter

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## Definition ( **[Miranda jr. '03]** )

We say that  $f \in L^1(X, m)$  has bounded variation if  $|Df|(X) < \infty$ . In that case  $|Df|$  is the restriction to open sets of a finite Borel measure.

$E \subset X$  has finite perimeter if  $\chi_E$  has bounded variation.

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Given a set of finite perimeter  $E \subset \mathbb{R}^n$ , the *reduced boundary*  $\mathcal{F}E$  is the set of those  $x \in \mathbb{R}^n$  such that there exists

$$\nu_E(x) := \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{and } |\nu_E(x)| = 1.$$

By Besicovitch's theorem,  $|D\chi_E|$  is concentrated on  $\mathcal{F}E$  and  $D\chi_E = \nu_E |D\chi_E|$ .

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Open problems

*Let  $E \subset \mathbb{R}^n$  be a set of finite perimeter. Then:*

*There exists a constant  $\epsilon = \epsilon(K, N) > 0$  such that if  $\text{Per}(E) \leq \epsilon$ , then  $E$  is approximately a hyperplane.*

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## Theorem ([De Giorgi '55])

Let  $E \subset \mathbb{R}^n$  be a set of finite perimeter. Then:

- for any  $x \in \mathcal{F}E$ , the rescaled sets  $E_{x,r} = \frac{E-x}{r}$  converge as  $r \downarrow 0$  in  $L^1_{loc}(\mathbb{R}^n)$  to the half-space

$$H = \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\};$$

- for any  $x \in \mathcal{F}E$  it holds

$$\lim_{r \rightarrow 0} \frac{|D\chi_E|(B_r(x))}{\omega_n r^{n-1}} = 1;$$

- $\mathcal{F}E$  is  $(\mathcal{H}^{n-1}, n-1)$ -rectifiable and  $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$ .

## Question

Is it possible to formulate and prove an analogue of De Giorgi's theorem in the RCD context?

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In [Ambrosio-Brué-S. '18] and [Brué-Pasqualetto-S. '19] we generalized De Giorgi's theorem to  $\text{RCD}(K, N)$  spaces.

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We obtain:

- existence of regular blow-ups almost everywhere w.r.t. the perimeter;
- almost-everywhere uniqueness of blow-ups and a representation formula for the perimeter measure;
- a representation formula for the perimeter measure.

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The analysis of blow-ups is the starting point in the Euclidean case.

Blow-ups of sets of finite perimeter in  $\text{RCD}(K, N)$  spaces

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Definition (Tangent to a set of finite perimeter)

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Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s. and  $E \subset X$  be a set of locally finite perimeter. We let  $\text{Tan}_x(X, d, m, E)$  be the collection of all quintuples  $(Y, \varrho, \mu, \gamma, F)$  such that

$(Y, \varrho, \mu, \gamma) \in \text{Tan}_x(X, d, m)$  and  $r_i \downarrow 0$  is a sequence of radii such that  $(X, r_i^{-1}d, m_{r_i}^x) \rightarrow (Y, \varrho, \mu, \gamma)$  in the pmGH sense;

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- $F$  is a set of locally finite perimeter in  $Y$  and  $\tilde{f} = \chi_E$ , considered as functions on the converging sequence of spaces above, converge in  $L^1_{\text{loc}}$  to  $\chi_F$ .

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Open problems

- There are lots of sets of finite perimeter, thanks to the coarea formula.
- The strategy via Besicovitch's theorem cannot be easily adapted.

In general, regular sets of finite perimeter turn into singular sets of finite perimeter.

The singular set is not necessarily regular, but the regularity of the boundary of a set of finite perimeter is not guaranteed.

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# Motivations & remarks

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of  $\text{RCD}(K, N)$   
spaces

Daniele Semola

Ricci curvature

$\text{RCD}(K, N)$   
spaces

Structure theory  
up to negligible  
sets

**De Giorgi's  
theorem on  
 $\text{RCD}(K, N)$   
spaces**

Open problems

- There are lots of sets of finite perimeter, thanks to the coarea formula.
- The strategy via Besicovitch's theorem cannot be easily adapted.

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## Example

Let  $(X, d, m)$  be an  $\text{RCD}(0, N)$  m.m.s. of finite measure,  $\mathbb{R} := (\mathbb{R}, d_{\text{eucl}}, \mathcal{L}^1)$  and let  $Y := X \times \mathbb{R}$  be the product space, with product metric and product measure.

If we let  $E := \{t \geq 0\}$ , then  $E$  is of finite perimeter and  $|D\chi_E|$  can be identified with  $m$ , up to identification of  $X$  with  $X \times \{0\}$ .

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Let  $E \subset X$  be of finite perimeter. Then for  $|D_{\chi_E}|$ -a.e.  $x \in X$  there exists a unique  $1 \leq k \leq n$  such that

$$\{(\mathbb{R}^k, d_{\text{euc}}, \mathcal{G}_k \mathcal{L}^k, \mathcal{D}^k, \{x_k > 0\})\} = \text{Tan}_x(X, d, \mathfrak{m}, E).$$

Let now  $\mathcal{F}_k E$  be the set of points with  $k$ -dimensional Euclidean blow-up.

For any set of finite perimeter  $E \subset X$  it holds that

$$|\mathcal{F}_k E| \leq \int \chi_k \, d|D_{\chi_E}|$$

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Theorem ( [Brué-Pasqualetto-S. '19] )

*Let  $E \subset X$  be of finite perimeter. Then for  $|D\chi_E|$ -a.e.  $x \in X$  there exists a unique  $1 \leq k \leq n$  such that*

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Let now  $\mathcal{F}_k E$  be the set of points with  $k$ -dimensional Euclidean blow-up.

For any set of finite perimeter  $E \subset X$  it holds that

$$|D\chi_E| = \sum_{k=1}^n c_k \mathcal{H}_k \llcorner \mathcal{F}_k E.$$

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Let  $E \subset X$  be of finite perimeter. Then for  $|D\chi_E|$ -a.e.  $x \in X$  there exists a unique  $1 \leq k \leq n$  such that

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Let now  $\mathcal{F}_k E$  be the set of points with  $k$ -dimensional Euclidean blow-up.

Corollary

For any set of finite perimeter  $E \subset X$  it holds that

$$|D\chi_E| = \sum_{k=1}^n c_k \mathcal{H}_{h_1} \llcorner \mathcal{F}_k E.$$

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*Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  m.m.s. and  $E \subset X$  a set of finite perimeter. Then, for any  $1 \leq k \leq n$ , it holds that  $\mathcal{F}_k E$  is  $(|D\chi_E|, k-1)$ -rectifiable.*

*Boundary of a set of finite perimeter has a good regularity regularity of the ambient spaces.*

The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

A key step is the construction of a suitable  $(k-1)$ -normal vector field.

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Boundaries of sets of finite perimeter have the same rectifiable regularity of the ambient spaces.

The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

Key words: Rectifiable sets, Geometric Measure Theory, Geometric Analysis

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A key step is the construction of a suitable *unit normal* vector field.

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Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  does it hold  $\dim X = \dim_{\mathbb{H}} X$ ?

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Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  with  $\dim X = n$  is there an open set  $U \subset X$  homeomorphic to a smooth  $n$ -dimensional manifold and with  $\mathfrak{m}(X \setminus U) = 0$ ?

These problems are open even for collapsed Ricci limits,  
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Open problems

Studying sets of finite perimeter:

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  with  $\dim X = n$  and a set of finite perimeter  $E \subset X$ , the tangent space to  $X$  has constant dimension (equal to  $n$ )  $|D\chi_E|$ -almost everywhere.

Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  space with  $\dim X = n$  and  $E \subset X$  be a set of finite perimeter. Then  $\mathcal{H}^n \llcorner E$  has  $n$ -finite  $\mathbb{R}^n$ -multiplicity and  $|D\chi_E| \ll \mathcal{H}^n \llcorner E$ .

# Open problems, II

Structure theory  
of  $\text{RCD}(K, N)$   
spaces

Daniele Semola

Ricci curvature

$\text{RCD}(K, N)$   
spaces

Structure theory  
up to negligible  
sets

De Giorgi's  
theorem on  
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spaces

Open problems

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Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  space with  $\dim X = n$ . Let  $E \subset X$  be a set of finite perimeter. Then  $\mathcal{F}_\nu E$  has  $\sigma$ -finite  $\mathcal{H}^{n-1}$  measure and  $|D\chi_E| \ll \mathcal{H}^{n-1} \llcorner \mathcal{F}_\nu E$ .

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**Open problems**

Thank you for your attention!