

Structure theory  
of  $\text{RCD}(K, N)$   
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Ricci curvature

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spaces

Structure theory  
up to negligible  
sets

De Giorgi's  
theorem on  
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Open problems

# Structure theory of spaces with lower Ricci bounds towards codimension one

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$\text{RCD}(K, N)$  spaces are a non smooth spaces with:

- Ricci curvature bounded from below by  $K$ ;
- dimension bounded from above by  $N$ ,

and looking infinitesimally like Riemannian manifolds (rather than Finsler ones).

Developing a theory of codimension one hypersurfaces in this non smooth framework.

Relying on the theory of sets of finite perimeter, pioneered by Caccioppoli and De Giorgi.

• A theory independent of specific analytical assumptions?

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$\text{RCD}(K, N)$  spaces are a **non smooth** spaces with:

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- **dimension** bounded from **above** by  $N$ ,

and looking infinitesimally like **Riemannian** manifolds (rather than Finsler ones).

Developing a theory of **sets of finite perimeter** on these spaces in the non-smooth framework.

Relying on the theory of **sets of finite perimeter**, pioneered by **Caccioppoli** and **De Giorgi**.

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and looking infinitesimally like **Riemannian** manifolds (rather than Finsler ones).

They are **not** necessarily **smooth** manifolds.

Relying on the theory of **sets of finite perimeter**, pioneered by **Caccioppoli** and **De Giorgi**.

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*... a theory independent of special topological investigations!*



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<sup>1</sup>from the review of L. C. Young to **[Caccioppoli '51]**.

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- **E. Brué, E. Pasqualetto**, D. Semola; Rectifiability of  $\text{RCD}(K, N)$  spaces via  $\delta$  splitting maps, accepted by Ann. Acad. Sci. Fenn. Math., (2020),
- L. Ambrosio, E. Brué, D. Semola; Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces. Geom. Funct. Anal. (2019),
- E. Brué, E. Pasqualetto, D. Semola; Rectifiability of the reduced boundary for sets of finite perimeter over  $\text{RCD}(K, N)$  spaces, preprint (2019).

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Open problems

Let us consider a smooth Riemannian manifold  $(M, g)$ ,  $p \in M$ ,  $u, v \in T_p M$  orthonormal vectors.

The sectional curvature  $K(u, v)$  governs distortions of distances

$$d(\exp_p(tu), \exp_p(tv)) = \sqrt{2}t \left( 1 - \frac{K(u, v)}{12}t^2 + O(t^3) \right) \quad \text{as } t \rightarrow 0.$$

Ricci curvature is obtained averaging the sectional curvatures:

$$\text{Ric}(u, u) := \sum_{i=2}^n K(u, e_i).$$

What is the Ricci curvature of  $\mathbb{Q}^n$ ?

What is the Ricci curvature of  $\mathbb{R}^n$  with the standard metric?

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In harmonic coordinates  $(x^i)$ :

$$\text{Ric}_i = -\frac{1}{2} \Delta g_i + \text{lower order terms.}$$

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# Ricci curvature: Lagrangian vs Eulerian

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Consider a smooth map  $\psi : M \rightarrow \mathbb{R}$  and let

$$T_t(x) := \exp(t\nabla\psi(x)).$$

Then if  $\dot{\gamma} := \frac{d}{dt}T_t(x)$  and  $\mathcal{J}(t) = \det DT_t(x)$  is the volume element,

$$\frac{d^2}{dt^2}(\mathcal{J}(t))^{1/n} + \frac{\text{Ric}(\dot{\gamma}, \dot{\gamma})}{n} \mathcal{J}^{1/n} \leq 0 \quad (1)$$

and

$$\Delta \frac{|\nabla\psi|^2}{2} - \nabla\psi \cdot \nabla\Delta\psi \geq \frac{(\Delta\psi)^2}{n} + \text{Ric}(\nabla\psi, \nabla\psi). \quad (2)$$

(1) is a Lagrangian formulation of (2) (the Eulerian formulation) and is more natural in the context of Ricci curvature bounds.

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(1) is a Lagrangian perspective and (2) (the Bochner inequality) is a dual Eulerian perspective on Ricci curvature.

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Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov volume inequality on monotonicity of volume ratios;
- Bishop-Gromov isoperimetric inequality;
- Bonnet-Myer lower bound;
- Bishop-Croke upper bound on isoperimetric inequality;

•  $\mathbb{R}^n$  is the only  $RCD(0, n)$  space with bounded diameter (Bishop-Croke);

•  $\mathbb{R}^n$  is the only  $RCD(0, n)$  space with bounded diameter and bounded Ricci curvature (Croke-Whitney-Young);

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Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- Bishop-Gromov inequality on monotonicity of volume ratios;

- volume comparison for manifolds with lower Ricci curvature;

- Cheeger-Croke isoperimetric inequality;

- volume growth of manifolds with lower Ricci curvature.

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- **Bishop-Gromov** inequality on monotonicity of volume ratios;
- **Cheeger-Gromoll** splitting theorem;
- **Li-Yau** heat kernel bounds;
- **Lévy-Gromov** isoperimetric inequality.

## Theorem ([Gromov '82])

*The class  $\mathcal{M}_{N,D,K}$  of smooth Riemannian manifolds with dimension  $N$ , diameter bounded above by  $D$  and Ricci curvature bounded below by  $K$  is precompact w.r.t. the Gromov-Hausdorff topology.*

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## Remark

Any manifold can be endowed with a Riemannian metric with Ricci curvature bounded above.

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields.

- **Bishop-Gromov** inequality on monotonicity of volume ratios;
- **Cheeger-Gromoll** splitting theorem;
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- **Lévy-Gromov** isoperimetric inequality.

Theorem ([Gromov '82])

*The class  $\mathcal{M}_{N,D,K}$  of smooth Riemannian manifolds with dimension  $N$ , diameter bounded above by  $D$  and Ricci curvature bounded below by  $K$  is precompact w.r.t. the Gromov-Hausdorff topology.*

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How do Riemannian manifolds in  $\mathcal{M}_{N,D,K}$  look like?

The question motivated the theory of Ricci limits, limits in the (pm)GH topology of manifolds in  $\mathcal{M}_{N,D,K}$ , initiated by Cheeger-Colding in the Nineties.

The study of Ricci limits improves our knowledge of  $\mathcal{M}_{N,D,K}$ :

- Riemannian manifolds with bounded Ricci curvature and volume bounded below enjoy uniform  $L^q$ -bounds for the Sobolev constants (Ricci curvature  $\geq -K$ ).
- The gradients of harmonic functions do not vary uniformly with respect to the base manifold.
- The Hausdorff dimension is bounded.

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The **synthetic** treatment of lower Ricci curvature bounds stems from the following

Do Ricci limit spaces have Ricci curvature bounded from below? In which sense?

Synthetic means not depending on the existence of a smooth structure, nor making reference to any notion of smoothness.

Can you give the theory of Ricci curvature bounds in a synthetic way?

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(cf. Branner '87): such a theory should deal with metric measure spaces. The heat flow could play a role.

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## Remark

Analogy with the theory of **Alexandrov spaces**, based on Toponogov's triangle comparison.

- [Gromov '81]: such a theory should deal with **metric measure spaces**. The **heat flow** could play a role;
- [Cheeger-Colding '97]: necessity to localize the Bishop-Gromov volume monotonicity in single directions.

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After several contributions:

- [McCann '97], [Otto-Villani '00], [Cordero-Erausquin-McCann-Schmuckenschläger '01], [Sturm-Von Renesse '07], for the connections between Optimal Transport and Ricci curvature on Riemannian manifolds;

• [Colding-De Lellis '11], [De Lellis-Cesari '12], for the proposal of the RCD( $K, N$ ) synthetic curvature condition on metric measure spaces;

• [De Lellis-Cesari '12], [De Lellis-Cesari '13], [De Lellis-Cesari '14], [De Lellis-Cesari '15], [De Lellis-Cesari '16], [De Lellis-Cesari '17], [De Lellis-Cesari '18], [De Lellis-Cesari '19], [De Lellis-Cesari '20], for the RCD( $K, N$ ) structure theorem;

• [De Lellis-Cesari '13], [De Lellis-Cesari '14], [De Lellis-Cesari '15], [De Lellis-Cesari '16], [De Lellis-Cesari '17], [De Lellis-Cesari '18], [De Lellis-Cesari '19], [De Lellis-Cesari '20], for the RCD( $K, N$ ) volume comparison;

• [De Lellis-Cesari '13], [De Lellis-Cesari '14], [De Lellis-Cesari '15], [De Lellis-Cesari '16], [De Lellis-Cesari '17], [De Lellis-Cesari '18], [De Lellis-Cesari '19], [De Lellis-Cesari '20], for the RCD( $K, N$ ) perimeter theory.

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- [Sturm '06] and [Lott-Villani '07], for the proposal of the CD( $K, N$ ) Curvature-Dimension condition on metric measure spaces;

[Ambrosio-Colesanti-Mantegani '05], [Colesanti-Mantegani '07], [Ambrosio-Colesanti '07], [Ambrosio-Mantegani '09], [Ambrosio-Mantegani '10], [Ambrosio-Mantegani '11], [Ambrosio-Mantegani '12], [Ambrosio-Mantegani '13], [Ambrosio-Mantegani '14], [Ambrosio-Mantegani '15], [Ambrosio-Mantegani '16], [Ambrosio-Mantegani '17], [Ambrosio-Mantegani '18], [Ambrosio-Mantegani '19]

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For any  $K \in \mathbb{R}$  and  $1 \leq N < \infty$  we say that  $(X, d, m)$  is RCD( $K, N$ ) if it is an infinitesimally Hilbertian CD( $K, N$ ) m.m.s.

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After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

A space  $(X, d, \mu)$  is  $\text{RCD}(K, N)$  if and only if

it is a metric measure space with finite measure, bounded diameter, bounded curvature, bounded dimension, and bounded Ricci curvature.

Moreover, the space is a Riemannian manifold with bounded Ricci curvature and bounded dimension.

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## Theorem $(\text{RCD}^*(K, N)$ spaces)

*A m.m.s.  $(X, d, m)$  is  $\text{RCD}^*(K, N)$  if:*

*$m(B_r(x)) \leq c_1 \exp(c_2 r^2)$  for some  $x \in X$  and constants  $c_1, c_2 > 0$ ;*

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*•  $m(B_r(x)) \leq c_1 \exp(c_2 r^2)$  for some  $x \in X$  and constants  $c_1, c_2 > 0$ ;*

*• it is infinitesimally Hilbertian ( $W^{1,2}(X, d, m)$  is Hilbert);*

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Open problems

After [Bacher-Sturm '10], [Erbar-Kuwada-Sturm '15] and [Ambrosio-Mondino-Savaré '15], inspired by the theory of Bakry-Émery-Ledoux, we have:

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## Remark

$\text{RCD}^*(K, N)$  is equivalent to  $\text{RCD}(K, N)$  if  $m(X) < \infty$ , thanks to [Cavalletti-Milman '16].

# Remarks & motivations

## Structure theory of $RCD(K, N)$ spaces

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Open problems

- The splitting theorem ([Cheeger-Colding], [Cheeger-Colding]) holds for  $RCD(0, N)$  spaces [Gigli '10]. The split factor is  $RCD(0, N - 1)$ .
- cones over  $RCD(N - 2, N - 1)$  spaces are  $RCD(0, N)$  [Ketterer '13], quotients of  $RCD^*(K, N)$  spaces under suitable group actions are  $RCD^*(K, N)$  spaces, [Galaz-García-Kell-Mondino-Sosa '17];
- applications of methods in [Gigli '10] are also in [Gigli '12];
- there are some open problems related to the structure theory of  $RCD(K, N)$  spaces, see [Semola '17] and [Semola '18];
- $RCD(K, N - 1)$  spaces are  $RCD(K, N)$  spaces, but the converse does not hold, see [Gigli '10].
- there are open problems related to the structure theory of  $RCD(K, N)$  spaces, see [Semola '17] and [Semola '18].

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- pmGH limits of manifolds in  $\mathcal{M}_{N,D,K}$  are  $\text{RCD}(K, N)$  spaces;
- Alexandrov spaces with dimension  $n$  and curvature bounded below by  $k$  equipped with the Hausdorff measure  $\mathcal{H}^n$  are  $\text{RCD}(k(n - 1), n)$  spaces [Petrunin '11], [Zhang-Zhu '10];
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# Tangent spaces

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How regular is an  $\text{RCD}(K, N)$  space?

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  and  $x \in X$  we let  
 $\text{Tan}_x(X, d, m)$  be the set of all pmGH limits

$$(Y, d_Y, m_Y, \gamma) = \lim_{r \rightarrow \infty} (X, r^{-1}d, m_r^x, x),$$

where  $r \downarrow 0$  and  $m_r^x = c_r^x m$  for some normalizing constant  $c_r^x > 0$ .

If  $(X, d, m) = (M^n, g, m_g)$ , then

$$\text{Tan}_x(X, d, m) = \{(M^n, g_{x,r}, c_r^x m_g)\} \text{ for any } r \in \mathbb{R}^+$$

and the limit converges to a metric space if and only if the space is flat.



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Open problems

## Question

How regular is an  $\text{RCD}(K, N)$  space?

## Definition (Tangent cone)

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  and  $x \in X$  we let  $\text{Tan}_x(X, d, m)$  be the set of all pmGH limits

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- If  $(X, d, m) = (M^n, d_g, \text{vol}_g)$ , then  $\text{Tan}_x(X, d, m) = \{(\mathbb{R}^n, d_{\text{eucl}}, c_n \mathcal{L}^n, 0^n)\}$  for any  $x \in X$ ;
- the tangent cone to a metric cone at its tip is the cone itself.

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# Structure theory up to measure zero

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Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. For any  $1 \leq k \leq N$  let

$$R_k := \{x \in X : \text{Tan}_x(X, d, m) = \{(\mathbb{R}^k, d_{\text{euc}}, c_x \llcorner \mathcal{L}^k, 0^k)\}\}.$$

After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

$\mathcal{H}^k \llcorner R_k = 0$  for any  $1 \leq k \leq N$  and  $\mathcal{H}^k \llcorner R_k = 0$  for any  $1 \leq k \leq N$

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Open problems

## Definition ( $k$ -regular set)

Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  m.m.s.. For any  $1 \leq k \leq N$  let

$$\mathcal{R}_k := \left\{ x \in X : \text{Tan}_x(X, d, \mathfrak{m}) = \left\{ (\mathbb{R}^k, d_{\text{eucl}}, c_k \mathcal{L}^k, 0^k) \right\} \right\}.$$

After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

*It holds*

$$\mathfrak{m} \left( X \setminus \bigcup_{k=1}^N \mathcal{R}_k \right) = 0.$$

*Furthermore, for any  $1 \leq k \leq N$  the  $k$ -regular set  $\mathcal{R}_k$  is  $(\mathfrak{m}, k)$ -rectifiable and  $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$ , for some  $\theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$ .*

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$$\mathcal{R}_k := \left\{ x \in X : \text{Tan}_x(X, d, \mathfrak{m}) = \left\{ (\mathbb{R}^k, d_{\text{eucl}}, c_k \mathcal{L}^k, 0^k) \right\} \right\}.$$

After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

*It holds*

$$\mathfrak{m} \left( X \setminus \bigcup_{k=1}^N \mathcal{R}_k \right) = 0.$$

*Furthermore, for any  $1 \leq k \leq N$  the  $k$ -regular set  $\mathcal{R}_k$  is  $(\mathfrak{m}, k)$ -rectifiable and  $\mathfrak{m} \llcorner \mathcal{R}_k = \theta \mathcal{H}^k$ , for some  $\theta \in L^1_{\text{loc}}(\mathcal{H}^k \llcorner \mathcal{R}_k)$ .*

# Structure theory up to measure zero

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spaces

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Ricci curvature

$\text{RCD}(K, N)$   
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theorem on  
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After [Mondino-Naber '14], [Kell-Mondino '16], [De Philippis-Marchese-Rindler '16] and [Gigli-Pasqualetto '16]:

## Theorem

*It holds*

$$m \left( X \setminus \bigcup_{k=1}^{\lfloor N \rfloor} \mathcal{R}_k \right) = 0.$$

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# Constancy of the dimension

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*Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s.. Then there exists  $1 \leq n \leq N$  such that*

$$m(X \setminus \mathcal{R}_n) = 0.$$

*This strategy has been recently generalized to the RCD framework in [Brué-S. '18].*

In [Brué-S. '18] we obtain new regularity estimates for flows of Sobolev vector fields, valid beyond the case needed for proving constancy of the dimension.

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# Structure theory via harmonic $\delta$ -splitting maps

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They have played a fundamental role in the Cheeger-Colding-Naber theory of Ricci limits.

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## Definition (Harmonic $\delta$ -splitting map)

Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  m.m.s. and  $B_r(x) \subset X$ . We say that  $u : B_r(x) \rightarrow \mathbb{R}^k$  is a  $\delta$ -splitting map provided:

1.  $u$  has harmonic and Lipschitz components;

2.  $u$  is  $\delta$ -close to a harmonic Lipschitz map.

They have played a fundamental role in the **Cheeger-Colding-Naber** theory of Ricci limits.



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- $u$  has harmonic and Lipschitz components;

- $\frac{\int_{B_r(x)} |\text{Hess} u|^2 dm}{m(B_r(x))} \leq \delta;$

- $\frac{\int_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| dm}{m(B_r(x))} \leq \delta.$

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They have played a fundamental role in the **Cheeger-Colding-Naber** theory of Ricci limits.

In [Brué-Pasqualetto-S. 20] we gave simplified proofs of the structure theorems for  $\text{RCD}(K, N)$  spaces using harmonic  $\delta$ -splitting maps to rectify.

# Structure theory via harmonic $\delta$ -splitting maps

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## Harmonic $\delta$ -splitting maps and structure theory

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So far we have a good understanding of the structure of  $\text{RCD}(K, N)$  spaces up to  $m$ -negligible sets.

Pushing the study of  $\text{RCD}(K, N)$  spaces up to codimension one subsets.

In the last couple of years:

• the rectifiability of  $\text{RCD}(K, N)$  spaces up to  $m$ -negligible sets

• the structure of boundary of non collapsed  $\text{RCD}$  spaces

• the structure of boundary of collapsed  $\text{RCD}$  spaces

• the structure of boundary of  $\text{RCD}$  spaces with boundary

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Pushing the study of  $\text{RCD}(K, N)$  spaces up to codimension one subsets.

In the last couple of years:

• [Colding and De Lellis](#) proved that the singular set of a  $\text{RCD}(K, N)$  space is  $(N-2)$ -negligible.

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## Aim

Pushing the study of  $\text{RCD}(K, N)$  spaces up to codimension one subsets.

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In the last couple of years:

- rectifiability of singular sets on non collapsed Ricci limits  
(Cheeger-Jiang-Naber '18);

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## Aim

Pushing the study of  $\text{RCD}(K, N)$  spaces up to codimension one subsets.

In the last couple of years:

- rectifiability of singular sets on non collapsed Ricci limits (Chen-Jiang Paper '18);
- notions of boundary on non collapsed RCD spaces (De Philippis-Digl'18, Kapovich-Mondino '19);

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Can we get some information about the behaviour of an  $\text{RCD}(K, N)$  space along a "hypersurface"? What do we mean by hypersurface?

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Pushing the study of  $\text{RCD}(K, N)$  spaces up to codimension one subsets.

In the last couple of years:

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## Question

Can we get some information about the behaviour of an  $\text{RCD}(K, N)$  space along a “hypersurface”? What do we mean by hypersurface?



# Sets of finite perimeter

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Theory pioneered by **Caccioppoli** and **De Giorgi** in the 1950s.

On  $\mathbb{R}^n$ , sets of finite perimeter are Borel sets  $E$  such that the distributional derivative  $D\chi_E$  is a finite (vector valued) measure.

Let  $(X, d, m)$  be a m.m.s. and  $f \in L^1_{loc}(X, m)$ . Given an open set  $A \subset X$  we let

$$|Df|(A) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_A \text{lip} f_n \, dm : f_n \in \text{Lip}_{loc}(A), f_n \rightarrow f \text{ in } L^1_{loc}(A, m) \right\}.$$

# Sets of finite perimeter

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Theory pioneered by **Caccioppoli** and **De Giorgi** in the 1950s.

On  $\mathbb{R}^n$ , sets of finite perimeter are Borel sets  $E$  such that the distributional derivative  $D\chi_E$  is a finite (vector valued) measure.

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## Definition ( **[Miranda jr. '03]** )

We say that  $f \in L^1(X, m)$  has bounded variation if  $|Df|(X) < \infty$ . In that case  $|Df|$  is the restriction to open sets of a finite Borel measure.

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Given a set of finite perimeter  $E \subset \mathbb{R}^n$ , the *reduced boundary*  $\mathcal{F}E$  is the set of those  $x \in \mathbb{R}^n$  such that there exists

$$\nu_E(x) := \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{and } |\nu_E(x)| = 1.$$

By Besicovitch's theorem,  $|D\chi_E|$  is concentrated on  $\mathcal{F}E$  and  $D\chi_E = \nu_E |D\chi_E|$ .

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*Let  $E \subset \mathbb{R}^n$  be a set of finite perimeter. Then:*

*There exists a constant  $\epsilon = \epsilon(K, N) > 0$  such that if  $\text{Per}(E) < \epsilon$ , then  $E$  is approximately a hyperplane.*

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## Theorem ([De Giorgi '55])

Let  $E \subset \mathbb{R}^n$  be a set of finite perimeter. Then:

- for any  $x \in \mathcal{F}E$ , the rescaled sets  $E_{x,r} = \frac{E-x}{r}$  converge as  $r \downarrow 0$  in  $L^1_{loc}(\mathbb{R}^n)$  to the half-space

$$H = \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\};$$

- for any  $x \in \mathcal{F}E$  it holds

$$\lim_{r \rightarrow 0} \frac{|D\chi_E|(B_r(x))}{\omega_n r^{n-1}} = 1;$$

- $\mathcal{F}E$  is  $(\mathcal{H}^{n-1}, n-1)$ -rectifiable and  $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \mathcal{F}E$ .

## Question

Is it possible to formulate and prove an analogue of De Giorgi's theorem in the RCD context?

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- a representation formula for the perimeter measure.

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The analysis of blow-ups is the starting point in the Euclidean case.

Blow-ups of sets of finite perimeter in  $\text{RCD}(K, N)$  spaces

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Definition (Tangent to a set of finite perimeter)

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## Definition (Tangent to a set of finite perimeter)

Let  $(X, d, m)$  be an  $\text{RCD}(K, N)$  m.m.s. and  $E \subset X$  be a set of locally finite perimeter. We let  $\text{Tan}_x(X, d, m, E)$  be the collection of all quintuples  $(Y, \varrho, \mu, \gamma, F)$  such that

$(Y, \varrho, \mu, \gamma) \in \text{Tan}_x(X, d, m)$  and  $r_i \downarrow 0$  is a sequence of radii such that  $(X, r_i^{-1}d, m_{r_i}^x) \rightarrow (Y, \varrho, \mu, \gamma)$  in the pmGH sense;

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- $(Y, \varrho, \mu, \gamma) \in \text{Tan}_x(X, d, m)$  and  $r_i \downarrow 0$  is a sequence of radii such that  $(X, r_i^{-1}d, m_{r_i}^x) \rightarrow (Y, \varrho, \mu, \gamma)$  in the pmGH sense;
- $F$  is a set of locally finite perimeter in  $Y$  and  $i = \chi_E$ , considered as functions on the converging sequence of spaces above, converge in  $L^1_{\text{loc}}$  to  $\chi_F$ .

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- $(Y, \varrho, \mu, \gamma) \in \text{Tan}_x(X, d, m)$  and  $r_i \downarrow 0$  is a sequence of radii such that  $(X, r_i^{-1}d, m_{X}^{r_i}, x) \rightarrow (Y, \varrho, \mu, \gamma)$  in the pmGH sense;
- $F$  is a set of locally finite perimeter in  $Y$  and  $f_i = \chi_E$ , considered as functions on the converging sequence of spaces above, converge in  $L^1_{loc}$  to  $\chi_F$ .

# Blow-ups of sets of finite perimeter

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Open problems

## Remark

The analysis of blow-ups is the starting point in the Euclidean case.

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Open problems

- There are lots of sets of finite perimeter, thanks to the coarea formula.
- The strategy via Besicovitch's theorem cannot be easily adapted.

In general, regular sets of finite perimeter turn into sets of finite perimeter with some regularity.

The regularity theory is based on the regularity of the level sets of the distance function.

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## Example

Let  $(X, d, m)$  be an  $\text{RCD}(0, N)$  m.m.s. of finite measure,  $\mathbb{R} := (\mathbb{R}, d_{\text{eucl}}, \mathcal{L}^1)$  and let  $Y := X \times \mathbb{R}$  be the product space, with product metric and product measure.

If we let  $E := \{t \geq 0\}$ , then  $E$  is of finite perimeter and  $|D\chi_E|$  can be identified with  $m$ , up to identification of  $X$  with  $X \times \{0\}$ .

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- In this way results about sets of finite perimeter turn into results about  $\text{RCD}(K, N)$  spaces.
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Let  $E \subset X$  be of finite perimeter. Then for  $|D_{\chi_E}|$ -a.e.  $x \in X$  there exists a unique  $1 \leq k \leq n$  such that

$$\{(\mathbb{R}^k, d_{\text{euc}}, \mathcal{G}_k \mathcal{L}^k, \mathcal{D}^k, \{x_k > 0\})\} = \text{Tan}_x(X, d, \mathfrak{m}, E).$$

Let now  $\mathcal{F}_k E$  be the set of points with  $k$ -dimensional Euclidean blow-up.

For any set of finite perimeter  $E \subset X$  it holds that

$$|\mathcal{F}_k E| \leq \int \chi_k \, d|D_{\chi_E}|$$

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Theorem ( [Brué-Pasqualetto-S. '19] )

Let  $E \subset X$  be of finite perimeter. Then for  $|D\chi_E|$ -a.e.  $x \in X$  there exists a unique  $1 \leq k \leq n$  such that

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Let now  $\mathcal{F}_k E$  be the set of points with  $k$ -dimensional Euclidean blow-up.

For any set of finite perimeter  $E \subset X$  it holds that

$$|D\chi_E| = \sum_{k=1}^n c_k \mathcal{H}_k \llcorner \mathcal{F}_k E.$$

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Theorem ( [\[Brué-Pasqualetto-S. '19\]](#) )

Let  $E \subset X$  be of finite perimeter. Then for  $|D\chi_E|$ -a.e.  $x \in X$  there exists a unique  $1 \leq k \leq n$  such that

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Let now  $\mathcal{F}_k E$  be the set of points with  $k$ -dimensional Euclidean blow-up.

Corollary

For any set of finite perimeter  $E \subset X$  it holds that

$$|D\chi_E| = \sum_{k=1}^n c_k \mathcal{H}_{h_1} \llcorner \mathcal{F}_k E.$$

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*Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  m.m.s. and  $E \subset X$  a set of finite perimeter. Then, for any  $1 \leq k \leq n$ , it holds that  $\mathcal{F}_k E$  is  $(|D\chi_E|, k-1)$ -rectifiable.*

*Standard tools used in this proof include: the volume growth regularity of the ambient spaces.*

The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

A key step is the construction of a suitable  $(k-1)$ -normal vector field.

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Boundaries of sets of finite perimeter have the same rectifiable regularity of the ambient spaces.

The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

Key words: Rectifiable sets, reduced boundary, perimeter, De Giorgi's theorem

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Remark

Boundaries of sets of finite perimeter have the same rectifiable regularity of the ambient spaces.

The proof combines tools from Functional Analysis, Geometric Analysis and Geometric Measure Theory.

A key step is the construction of a suitable unit normal vector field.

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A key step is the construction of a suitable *unit normal* vector field.

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Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  we let  $\dim X$  be the unique natural  $1 \leq n \leq N$  such that  $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$ .

**Open problem**

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  does it hold  
 $\dim X = \dim_{\mathbb{H}} X$ ?

**Open problem**

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  with  $\dim X = n$  is there an open set  $U \subset X$  homeomorphic to a smooth  $n$ -dimensional manifold and with  $\mathfrak{m}(X \setminus U) = 0$ ?

These problems are open even for collapsed Ricci limits,  
[Colding-Naber '12], [Naber '14, '20].

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Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, m)$  we let  $\dim X$  be the unique natural  $1 \leq n \leq N$  such that  $m(X \setminus \mathcal{R}_n) = 0$ .

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Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  with  $\dim X = n$ , is there an open set  $U \subset X$  homeomorphic to a smooth  $n$ -dimensional manifold and with  $\mathfrak{m}(X \setminus U) = 0$ ?

These problems are open even for collapsed Ricci limits,  
[Colding-Naber '12], [Naber '14, '20].



# Open problems, I

Structure theory  
of  $\text{RCD}(K, N)$   
spaces

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Ricci curvature

$\text{RCD}(K, N)$   
spaces

Structure theory  
up to negligible  
sets

De Giorgi's  
theorem on  
 $\text{RCD}(K, N)$   
spaces

Open problems

## Definition

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  we let  $\dim X$  be the unique natural  $1 \leq n \leq N$  such that  $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$ .

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Open problems

Studying sets of finite perimeter:

Given an  $\text{RCD}(K, N)$  m.m.s.  $(X, d, \mathfrak{m})$  with  $\dim X = n$  and a set of finite perimeter  $E \subset X$ , the tangent space to  $X$  has constant dimension (equal to  $n$ )  $|D_{\chi_E}|$ -almost everywhere.

Let  $(X, d, \mathfrak{m})$  be an  $\text{RCD}(K, N)$  space with  $\dim X = n$ . Let  $E \subset X$  be a set of finite perimeter. Then  $\mathcal{H}^n \llcorner E$  has  $n$ -finite  $\mathbb{R}^n$ -multiplicity and  $|D_{\chi_E}| \ll \mathcal{H}^n \llcorner E$ .

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**Open problems**

Thank you for your attention!